ON THE LOCI OF THE LINES INCIDENT WITH k $(r-2)$-SPACES IN S_r

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The problem of the determination of the number N_r of lines that meet $2r-2$ given $(r-2)$-spaces in S_r has been solved.* Schubert's symbolic or enumerative method is powerful for the solution of problems of this kind and has indeed been the one used, but it does not offer any insight into the nature of the geometry involved. It is the purpose of this paper to re-determine the number N_r and also to obtain the loci of the ∞^{2r-2-k} lines incident with k given $(r-2)$-spaces in S_r, where $r<k\leqq 2r-2$.

For our purpose we make use of the known theorem:† *The locus of the ∞^{r-2} lines incident with r general $(r-2)$-spaces in S_r is a hypersurface V_{r-1}^{r-1}.*

Now consider r of the given $(r-2)$-spaces, say $S_{r-2}^{(i)}$ $[i=1, 2, \cdots, r]$. They yield a V_{r-1}^{r-1} whose generators are incident with them. Any of the remaining $k-r$ given $(r-2)$-spaces, say $S_{r-2}^{(r+1)}$, meets V_{r-1}^{r-1} in a V_{r-3}^{r-1}. Any hyperplane S'_{r-1} through $S_{r-2}^{(r+1)}$ meets $S_{r-2}^{(i)}$ in r $(r-3)$-spaces. The ∞^{r-4} lines that meet these r $(r-3)$-spaces are in S'_{r-1} and hence meet $S_{r-2}^{(r+1)}$, and they form a $V_{r-3}^{M_r'}$ whose order M_r' is to be determined later. Hence the ∞^{r-3} lines incident with $r+1$ $(r-2)$-spaces in S_r form a $V_{r-2}^{M_r+M_r'}$ (writing M_r for $r-1$), for it is met by S'_{r-1} in a $V_{r-3}^{M_r+M_r'}\equiv V_{r-3}^{M_r}+V_{r-3}^{M_r'}$.

Now the $(r+2)$th $(r-2)$-space, $S_{r-2}^{(r+2)}$, meets $V_{r-3}^{M_r+M_r'}$ in a $V_{r-4}^{M_r+M_r'}$. A hyperplane S''_{r-1} through $S_{r-2}^{(r+2)}$ intersects the other $r+1$ $(r-2)$-spaces in $r+1$ $(r-3)$-spaces and the ∞^{r-5} lines incident with the latter form a $V_{r-4}^{M_r''}$. Hence the

* See C. Segre, *Mehrdimensionale Räume*, Encyklopädie der Mathematischen Wissenschaften, vol. III: 2, pp. 813, 814 where full references are given.

† This Bulletin, vol. 32, No. 5, pp. 553–554.

∞^{r-4} lines incident with $r+2$ $(r-2)$-spaces in S_r form a $V_{r-3}^{M_r+M_r'+M_r''}$, for it is met by S_{r-1}'' in a $V_{r-4}^{M_r+M_r'+M_r''}\equiv V_{r-4}^{M_r+M_r'}+V_{r-4}^{M_r''}$.

Continuing this process of reasoning we soon come to $S_{r-2}^{(k)}$, the kth or the last of the given $(r-2)$-spaces, and find that the ∞^{2r-k-2} lines meeting k $(r-2)$-spaces in S_r form a $V_{2r-k-1}^{M_r+M_r'+\cdots+M_r^{(k-r)}}$, for it is intersected by a hyperplane S_{r-1} through $S_{r-2}^{(k)}$ in a $V_{2r-k-2}^{M_r+M_r'+\cdots+M_r^{(k-r)}}$ composed of a $V_{2r-k-2}^{M_r+M_r'+\cdots+M_r^{(k-r-1)}}$ and a $V_{2r-k-2}^{M_r^{(k-r)}}$. The former is contained in $S_{r-2}^{(k)}$ and the latter is the locus of the ∞^{2r-k-3} lines incident with the $k-1$ $(r-3)$-spaces in which S_{r-1} meets the other $k-1$ $(r-2)$-spaces supposed to have already been considered.

If $k=2r-3$, the ∞^1 lines incident with $2r-3$ $(r-2)$-spaces form a $V_2^{N_r}$ where

$$N_r = \sum_{j=0}^{r-3} M_r^{(j)}. \tag{1}$$

Since a general $(r-2)$-space can have only N_r points in common with $V_2^{N_r}$, the required number of lines meeting $2r-2$ given $(r-2)$-spaces is N_r given by (1).

Now to determine the values of $M_r^{(j)}$. Replacing r by $r+1$ in (1), we have

$$N_{r+1} = \sum_{j=0}^{r-2} M_{r+1}^{(j)},$$

where

$$M_{r+1}^{(j)} = \sum_{h=0}^{j} M_r^{(h)}. \tag{2}$$

Then, from (1) and (2),

$$N_r = M_{r+1}^{(r-2)} = M_{r+1}^{(r-3)}. \tag{3}$$

Decreasing r by unity in (2) and writing out the resulting equalities for all values of j from zero to $r-3$, we obtain, without difficulty, remembering $M_r=r-1$,

$$M_r^{(j)} = \sum_{j=0}^{r-3} M_{j+3}^{(j-1)} = \frac{(r+j-1)!(r-j-2)}{(r-1)!(j+1)!} \tag{4}$$

and also

$$M_{r+1}^{(j)} = \frac{(r+j)!(r-j-1)}{r!(j+1)!}. \tag{5}$$

Putting $j=r-2$ or $r-3$, we have, from (3),

$$N_r = \frac{(2r-2)!}{r!(r-1)!},$$

the required formula for N_r in terms of r.

The locus of the ∞^{2r-k-2} lines that meet k given $(r-2)$-spaces has been obtained and its order is $\sum_{h=0}^{k-r} M_r^{(h)}$ and, on account of (2) and (5), this is equal to

$$M_{r+1}^{(k-r)} = \frac{k!(2r-k-1)}{r!(k-r-1)!}.$$

Attention is called to the fact that a general $(r-2)$-space meets this locus in a V_{2r-k-3} of the same order but there are exactly k, that is, the given, $(r-2)$-spaces each of which meets it in a V_{2r-k-2} of order

$$\sum_{h=0}^{k-r-1} M_r^{(h)} = M_{r+1}^{(k-r-1)} = \frac{(k-1)!(2r-k)}{r!(k-r)!}.$$

A hyperplane through one of these k $(r-2)$-spaces meets the locus again in a V_{2r-k-2} of order

$$M_r^{(k-r)} = \frac{(k-1)!(2r-k-2)}{(r-1)!(k-r+1)}.$$

It is to be noted that these formulas do not hold for $k=r$; for, if $k=r$, $M_r=r-1$, $M_{r+1}=r$.

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