which these principles are formulated, what precisely distinguishes a logical principle from an ordinary contingent proposition is the absence of alternative possibilities, this impossibility of alternatives being explicitly stated in the formulation of the principle itself; so that whoever holds with regard to an assigned proposition that there could be circumstances under which that proposition would fail is holding that the proposition in question is not a logical principle.

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NOTE ON EINSTEIN'S EQUATION OF AN ORBIT

BY JAMES PIERPONT

In a paper* bearing the above title Morley has given an extremely elegant solution of Einstein's equation

$$\left(\frac{dx}{d\theta}\right)^2 = 2x^3 - x^2 + 2\lambda x - \lambda^2(1 - e^2), \tag{1}$$

which defines the motion of a single planet about the sun. Here r, θ are the polar coordinates of the planet, a the major semi-axis, e the eccentricity of the orbit, M the mass of the sun, and

$$x = \frac{M}{r}, \quad \lambda = \frac{M}{a(1 - e^2)}\cdot \tag{2}$$

In Eddington units, $M = 1.45$. For Mercury, the values are

$$a = 5.8\cdot 10^{-7}, \qquad e = 0.206, \qquad \lambda = 2.6\cdot 10^{-8}.$$

The roots of the right side of (1) are thus, to a high degree of approximation,

$$(1 - e)\lambda, \qquad (1 + e)\lambda, \qquad \tfrac{1}{2} - 2\lambda.$$

* American Journal of Mathematics, vol. 43 (1921), p. 29. I notice two obvious typographical errors in this paper. In the last term of (2) α should be α^2; also just below, x_1 should read $x_1 = \frac{1}{2} - 2\alpha$.

We reduce (1) to Weierstrass' canonical form, on setting

$$x = p + \tfrac{1}{6}, \qquad d\theta = 2^{1/2} du.$$

This gives

$$\left(\frac{dp}{du}\right)^2 = 4(p - e_1)(p - e_2)(p - e_3),$$

where

(3) $e_1 = \frac{1}{3} - 2\lambda$, $e_2 = -\frac{1}{6} + \lambda(1 - e)$, $e_3 = -\frac{1}{6} + \lambda(1 + e)$,

the indices being chosen so that $e_2 < e_3 < e_1$.

We find now

$$\theta = 2^{1/2} \cdot t\omega_1, \qquad u = \omega_2 + t\omega_1.$$

To $t = 0$ corresponds maximum r, while to $t = 1$ corresponds minimum r. Thus as t increases from 0 to 1, the planet passes from aphelion to perihelion, and θ increases by $\Delta\theta = 2^{1/2}\omega_1$.

It only remains to calculate this quantity. Morley now remarks that "the appropriate formulas are given in works on the elliptic functions. But as the proofs of the full formulas are necessarily complicated, I shall interpolate a proof of the approximate formulas of a kind that is at once intelligible."

I wish to submit an alternate form of proof which requires only the simplest and most commonplace formulas.*

We have

$$\omega_1 = \frac{K}{(e_1 - e_2)^{1/2}}, \qquad (4), \text{ p. } 448,$$

$$k'^2 = \frac{e_1 - e_3}{e_1 - e_2}, \qquad (3), \text{ p. } 449,$$

$$q = \frac{1}{2}\,\frac{1 - k'^{1/2}}{1 + k'^{1/2}}, \qquad (3), \text{ p. } 437,$$

$$\left(\frac{2K}{\pi}\right)^{1/2} = 1 + 2q + 2q^4 + \cdots, \qquad (6), \text{ p. } 438.$$

* The numbers on the right refer to my book *Functions of a Complex Variable*. Unfortunately the most distressing confusion exists in the notations employed by different authors. For this reason I may be excused for referring to my book which contains only the briefest account of the elliptic functions. I take this occasion to correct formula (5), p. 448. The numerator in this formula should be $e_1 - e_2$ instead of $e_2 - e_1$.

Now, from (3),

$$e_1 - e_3 = \tfrac{1}{2}[1 - 2\lambda(3 + e)], \quad e_1 - e_2 = \tfrac{1}{2}[1 - 2\lambda(3 - e)],$$

$$k'^2 = [1 - 2\lambda(3 + e)][1 + 2\lambda(3 - e)] = 1 - 4\lambda e,$$

$$k'^{1/2} = 1 - \lambda e, \qquad q = \frac{1}{2}\,\frac{\lambda e}{2 - \lambda e} = \frac{1}{4}\lambda e,$$

$$\frac{2K}{\pi} = 1 + 4q, \qquad\qquad K = \pi\frac{1 + \lambda e}{2},$$

$$(e_1 - e_2)^{-1/2} = 2^{1/2}[1 + \lambda(3 - e)],$$

$$\omega_1 = \pi(1 + \lambda e)2^{1/2}\frac{1 + \lambda(3 - e)}{2} = \pi\frac{1 + 3\lambda}{2^{1/2}}.$$

Thus $\Delta\theta = \pi + 3\pi\lambda$. Hence the advance in the radius vector from perihelion to perihelion over 360° is $\phi = 6\pi\lambda$. Now in the Newtonian theory of elliptic motion $M = 4\pi^2 a^3/T^2$. Substituting in (2) and using one second as the unit of time, we find

$$\phi = \frac{24\pi^3 a^2}{c^2(1 - e^2)T^2}.$$

This is Einstein's celebrated formula for the advance of the perihelion of a planet.

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