$$(9)\qquad \begin{vmatrix} A & B & C \\ A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \end{vmatrix} = 0, \quad \begin{vmatrix} B_1 & C_1 \\ B_2 & C_2 \end{vmatrix} \cdot \begin{vmatrix} C_1 & A_1 \\ C_2 & A_2 \end{vmatrix} = \begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix}^2 \cdot O$$

represent the two eliminants of the system (8).

The two necessary and sufficient conditions to be fulfilled by three arbitrary functions $H(x, y)$, $K(x, y)$, and $R(x, y)$ of x and y in order that the congruence of circles

$$(\alpha - H)^2 + (\beta - K)^2 = R^2$$

which they determine in the (α, β)-plane may represent the derivative of a polygenic function are obtained by retransforming the two equations (9) *from the quantities u, v, M, N, O into the quantities x, y, H, K, R; the two final conditions contain R and the derivatives of H, K and R up to the third order.*

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ON THE INVERSION OF ANALYTIC TRANSFORMATIONS*

BY B. O. KOOPMAN†

We wish to consider the transformation

$$(1)\qquad x_i = f_i(y_1, \cdots, y_n), \qquad (i = 1, \cdots, n),$$

in the neighborhood of the origin $(y) = (0)$, at which point the functions f_i are analytic, and vanish simultaneously. We are interested in the case in which the jacobian

$$J = \frac{\partial(f_1, \cdots, f_n)}{\partial(y_1, \cdots, y_n)}$$

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The methods and point of view of the second chapter of W. F. Osgood's *Lehrbuch der Funktionentheorie* are assumed throughout this paper. Our results may be regarded as the extension of §20 of that work.

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is zero at the origin, without vanishing identically. We shall assume throughout that $(y)=(0)$ is a non-specialized point of the locus $J=0$; that is, it will be sufficient for our results to apply only when $(y)=(0)$ does not belong to certain exceptional $(2n-4)$-dimensional loci satisfying a further equation $E(y_1, \cdots, y_n)=0$, where E is analytic and relatively prime to J.

What we wish to accomplish is to obtain, with the aid of non-singular analytic transformations of the variables (x) and the variables (y), a standard form for the transformation (1), having particularly in view the investigation of its inverse.

Since $(y)=(0)$ is a non-specialized point of $J=0$, we shall have in a neighborhood of this point

$$J(y_1, \cdots, y_n) \equiv S(y_1, \cdots, y_n)^s \cdot L(y_1, \cdots, y_n),$$

where S and L are analytic at the origin, L does not vanish there, while S has there the value zero. Moreover, at least one derivative, say $\partial S/\partial y_1$, will be distinct from zero for $(y)=(0)$.

After the non-singular transformation

$$y_1' = S(y_1, \cdots, y_n), \qquad (y_2' = y_2, \cdots, y_n' = y_n),$$

the equations (1) retain the form

$$x_i = f_i'(y_1', \cdots, y_n'), \qquad (i = 1, \cdots, n), \tag{1'}$$

while their jacobian is readily found to be

$$J'(y_1', \cdots, y_n') \equiv y_1'^s \cdot L'(y_1', \cdots, y_n'),$$

L' being of the same nature as L. Hence we may and we shall assume that (1) has this form, that is, we shall drop the accents.

We now introduce, with Osgood, the matrix

$$\left\| \begin{matrix} \dfrac{\partial f_1}{\partial y_2}, & \cdots, & \dfrac{\partial f_n}{\partial y_2} \\ \cdot & \cdots & \cdot \\ \dfrac{\partial f_1}{\partial y_n}, & \cdots, & \dfrac{\partial f_n}{\partial y_n} \end{matrix} \right\|. \tag{2}$$

Suppose, in the first place, that when $y_1=0$, every element vanished identically. Considering the Taylor expansions of the f's, and observing that they are zero for $(y)=(0)$, we see that

$$f_i = y_1^{p_i}F_i(y_1, \cdots, y_n), \qquad (p_i > 0\,; i = 1, \cdots, n).$$

Here the F's are analytic at the origin, and are not identically zero when $y_1=0$. Since the origin is a non-specialized point, we assume that $F_1 \neq 0$ there. Then the transformation

$$\begin{cases} y_1' = y_1[F_1(y_1, \cdots, y_n)]^{1/p_1}, & \\ y_i' = y_i, & (i = 1, \cdots, n), \end{cases}$$

is analytic and non-singular at $(y)=(0)$; and we have

$$\begin{cases} f_1(y) = f_1'(y') = y_1'^{p_1}, & \\ f_i(y) = f_i'(y') = y_1'^{p_i}F_i(y_1', \cdots, y_n'), & (i = 2, \cdots, n). \end{cases}$$

Hence, dropping accents, we may assume (1) to have the form

$$(3) \qquad \begin{cases} x_1 = y_1^{p_1}, & \\ x_i = y_1^{p_i}F_i(y_1, \cdots, y_n), & (i = 2, \cdots, n). \end{cases}$$

In the case where the elements of (1) do not behave in the above manner, there will be a determinant j of (2) of order $r(0<r<n)$, which does not vanish identically when $y_1=0$, while every determinant in (2) of greater order is zero identically for $y_1=0$. Let us assume, as we may, that

$$j = \frac{\partial(f_{n-r+1}, \cdots, f_n)}{\partial(y_{n-r+1}, \cdots, y_n)}.$$

The origin being non-specialized, we assume that $j \neq 0$ when $(y)=(0)$.

The last r of the equations (1) admit the unique solutions

$$\begin{cases} y_{n-r+1} = g_{n-r+1}(y_1, \cdots, y_{n-r}\,; x_{n-r+1}, \cdots, x_n), \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ \qquad y_n = g_n(y_1, \cdots, y_{n-r}\,; x_{n-r+1}, \cdots, x_n), \end{cases}$$

where the g's are analytic for zero values of the arguments, and vanish for these values. When these expressions are substituted into the first $n-r$ equations (1), we obtain

$$\begin{aligned} x_i = \; & f_i[y_1, \cdots, y_{n-r}; g_{n-r+1}(y_1, \cdots, y_{n-r}; x_{n-r+1}, \\ & \cdots, x_n), \cdots, g_n(y_1, \cdots, y_{n-r}; x_{n-r+1}, \\ & \cdots, x_n)], \\ = \; & H_i(y_1, \cdots, y_{n-r}; x_{n-r+1}, \cdots, x_n), \\ & \qquad (i = 1, \cdots, n-r). \end{aligned} \tag{4}$$

I say that, when $y_1 = 0$, the H's do not contain the y's:

$$\frac{\partial H_i}{\partial y_j} \equiv 0 \text{ for } y_1 = 0; \quad (i = 1, \cdots, n-r; \; j = 1, \cdots, n-r).$$

This follows, as in the proof of the theorem of functional dependence, from the fact that, when $y_1 = 0$, all determinants of order $r+1$ of (2) are identically zero.* It follows that the equations (4) have the form

$$x_i = G_i(x_{n-r+1}, \cdots, x_n) + y_1^{p_i}\Gamma_i(y_1, \cdots, y_{n-r}; x_{n-r+1}, \cdots, x_n),$$

for $p_i > 0$; $i = 1, \cdots, n-r$, where G and Γ are analytic, and $G_i(0) = 0$.

Now perform the non-singular analytic transformation

$$\begin{cases} x_i' = x_i - G_i(x_{n-r+1}, \cdots, x_n), & (i = 1, \cdots, n-r), \\ x_j' = x_j, & (j = n-r+1, \cdots, n). \end{cases}$$

The equations (1) retain their form, becoming

$$x_i' = f_i'(y_1, \cdots, y_n), \qquad (i = 1, \cdots, n); \tag{1''}$$

and

$$J' = \frac{\partial(x_1', \cdots, x_n')}{\partial(y_1, \cdots, y_n)} = \frac{\partial(x_1', \cdots, x_n')}{\partial(x_1, \cdots, x_n)} \frac{\partial(x_1, \cdots, x_n)}{\partial(y_1, \cdots, y_n)},$$

* See Osgood, *Funktionentheorie*, vol. 2, Chap. 2, §23.

and hence $J'=J=y_1^sL$; also, when $y_1=0$, $j'=j$. Indeed, the properties which we have demonstrated for equation (1) are retained. But it is seen at once that $f_1', \cdots, f_{n-r}'$ all vanish identically when $y_1=0$. Hence, dropping the accents, we may write them in the form

$$f_i = y_1^{p_i}F_i(y_1, \cdots, y_n), \qquad (p_i > 0\,; i = 1, \cdots, n-r).$$

Reasoning as before, we may make the non-singular transformation

$$\begin{cases} y_1' = y_1[F_1(y_1, \cdots, y_n)]^{1/p_1}, & \\ y_i' = y_i, & (i = 2, \cdots, n), \end{cases}$$

and, dropping the accents, we may write the equations in the form

$$\begin{cases} x_1 = y_1^{p_1}, \\ x_2 = y_1^{p_2}F_2(y_1, \cdots, y_n), \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ x_{n-r} = y_1^{p_{n-r}}F_{n-r}(y_1, \cdots, y_n), \\ x_i = f_i(y_1, \cdots, y_n), \\ \qquad\qquad (i = n-r+1, \cdots, n). \end{cases}$$

A little computation shows that the new determinant j does not vanish for $(y)=(0)$. Hence we may apply the following non-singular transformation

$$\begin{cases} y_i' = y_i, & (i = 1, \cdots, n-r), \\ y_j' = f_j(y_1, \cdots, y_n), & (j = n-r+1, \cdots, n), \end{cases}$$

and, dropping the accents, we may write the equations in the form

$$(5)\qquad \begin{cases} x_1 = y_1^{p_1}, \\ x_2 = y_1^{p_2}\phi_2(y_1, \cdots, y_n), \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ x_{n-r} = y_1^{p_{n-r}}\phi_{n-r}(y_1, \cdots, y_n), \\ x_i = y_i, \qquad (i = n-r+1, \cdots, n). \end{cases}$$

These equations, which include (3) as a special case ($r=0$), will be made the basis of the rest of our work.

To begin with, we form the square matrix

$$
(6) \qquad \left\|\begin{matrix} \dfrac{\partial \phi_2}{\partial y_2} & \cdots & \dfrac{\partial \phi_{n-r}}{\partial y_2} \\ \cdot & \cdots & \cdot\ \cdot \\ \dfrac{\partial \phi_2}{\partial y_{n-r}} & \cdots & \dfrac{\partial \phi_{n-r}}{\partial y_{n-r}} \end{matrix}\right\|.
$$

Consider first the case where all the elements are identically zero when $y_1=0$. Then the ϕ's have the form

$$
\phi_i = \psi_i(y_{n-r+1}, \cdots, y_n) + y_1\phi_i'(y_1, \cdots, y_n), \qquad (i = 2, \cdots, n-r).
$$

In this case we form the matrix (6′), in which ϕ_i' replaces ϕ_i in (6). If all the elements of (6′) are identically zero for $y_1=0$, we repeat the process. Since the right-hand members of (5) actually contain *all* the y's (for $J \not\equiv 0$), we are eventually stopped; that is, we have, for a certain integer λ,

$$
\begin{cases} \phi_i = \psi_i + \psi_i' y_1 + \cdots + \psi_i^{(\lambda-1)} y_1^{\lambda-1} \\ \qquad\quad + y_1^{\lambda}\phi_i^{(\lambda)}(y_1, \cdots, y_n), \\ \psi_i^{(\kappa)} = \psi_i^{(\kappa)}(y_{n-r+1}, \cdots, y_n), \quad (i = 2, \cdots, n-r), \end{cases}
$$

where the matrix $(6^{(\lambda)})$ of the $\phi^{(\lambda)}$'s does not have all its elements identically zero when $y_1=0$.

In all cases, we are led to a matrix, (6) or $(6^{(\lambda)})$, in which at least one determinant δ of order $\rho(0<\rho<n-r)$ does not vanish identically for $y_1=0$, whereas every determinant of higher order is identically zero for that value. Let us assume, as we may, that

$$
\delta = \frac{\partial(\phi_{n-r-\rho+1}, \cdots, \phi_{n-r})}{\partial(y_{n-r-\rho+1}, \cdots, y_{n-r})},
$$

and that $\delta \neq 0$ at the non-specialized point $(y)=(0)$. We perform the non-singular substitution

$$
\begin{cases}
y_1' = y_1, \cdots, y_{n-r-\rho}' = y_{n-r-\rho}, \\
y_{n-r-\rho+1}' = \phi_{n-r-\rho+1}(y) - \phi_{n-r-\rho+1}(0), \\
\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
y_{n-r}' = \phi_{n-r}(y) - \phi_{n-r}(0), \\
y_{n-r+1}' = y_{n-r+1}, \cdots, y_n' = y_n.
\end{cases}
$$

This reduces (5) to a transformation which, dropping the accents, we may write in the form

$$
\begin{cases}
x_1 = y_1^{p_1}; \\
x_i = \omega_{i0} y_1^{p_i} + \omega_{i1} y_1^{p_i+1} + \cdots + \omega_{i,\lambda+\mu} y_1^{p_i+\lambda+\mu} + y_1^{p_i+\lambda+\mu}\Phi_i, \\
\qquad\qquad (i = 2, \cdots, n-r-\rho), \\
x_j = \omega_{j0} y_1^{p_j} + \omega_{j1} y_1^{p_j+1} + \cdots + \omega_{j\lambda} y_1^{p_j+\lambda} y_j, \\
\qquad\qquad (j = n-r-\rho+1, \cdots, n-r), \\
x_k = y_k, \qquad (k = n-r+1, \cdots, n).
\end{cases}
$$

Here $\omega_{i\beta} = \omega_{i\beta}(y_{n-r-\rho+1}, \cdots, y_n)$, $\omega_{j\beta} = \omega_{j\beta}(y_{n-r+1}, \cdots, y_n)$, when $\beta > 0$; otherwise, the ω's are constants. We repeat the discussion of the last two paragraphs, applying it to $\Phi_2, \cdots, \Phi_{n-r-\rho}$, and the matrix

$$
\begin{Vmatrix}
\dfrac{\partial \Phi_2}{\partial y_2} & \cdots & \dfrac{\partial \Phi_{n-r-\rho}}{\partial y_2} \\
\cdot & \cdots & \cdot \\
\dfrac{\partial \Phi_2}{\partial y_{n-r-\rho}} & \cdots & \dfrac{\partial \Phi_{n-r-\rho}}{\partial y_{n-r-\rho}}
\end{Vmatrix},
$$

in place of $\phi_2, \cdots, \phi_{n-r}$ and (6) or $(6^{(\lambda)})$. Continuing in this manner, we find, at the end of a finite number of steps, the final standard form for the equations (1) of the transformation we are considering:

$$
(7) \qquad
\begin{cases}
x_1 = y_1^{p_1}, \\
x_i = \sum_{q=0}^{\lambda_i} y_1^{p_i+q} \omega_{iq}(y_{i+1}, \cdots, y_n) + y_1^{p_i+\lambda_i} y_i, \\
\qquad\qquad (i = 2, \cdots, n-r), \\
x_j = y_j, \qquad (j = n-r+1, \cdots, n).
\end{cases}
$$

Here r may have any value from 0 to $n-1$. When $r=n-1$, the second group of equations is lacking in (7), and, when $r<n-1$,

$$\lambda_2 \geqq \lambda_3 \geqq \cdots \geqq \lambda_{n-r} \geqq 0.$$

The method of inversion of the transformation is obvious. After the first equation and the last r equations have been solved for y in terms of x, we can solve the last of the remainder, then the next to the last, and so on. We clearly find that the y's are functions of $(x_1^{1/p_1}, x_2, \cdots, x_n)$, meromorphic at the point $(0, 0, \cdots, 0)$.

It will be seen that the case treated by Osgood (loc. cit., "Case I") is the case where $r=n-1$. But the usefulness of the result for $r<n-1$ is in general less than when $r=n-1$, for the neighborhood of $(x)=(0)$ may be transformed by (7) into a portion of the $(y_1 \cdots y_n)$-space not lying in the neighborhood of $(y)=(0)$, (compare the transformation $x_1=y_1$, $x_2=y_1y_2$). When this is the case, the fact that the *neighborhoods* of the origins of the given $(y_1 \cdots y_n)$-space of (1) and the new $(y_1 \cdots y_n)$-space of (7) correspond is insufficient to furnish complete information regarding the inversion of (1). It may be necessary to apply these results to an infinite set of neighborhoods. Consider, for example, the transformation $x_1=y_1$, $x_2=y_1 \tan y_2$.

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