ON A CERTAIN SYSTEM OF ∞^{r-2} LINES IN r-SPACE

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This paper deals with the following theorem.

The locus of ∞^{r-2} *lines incident with* r *given* $(r-2)$*-spaces in* S_r *is an* $(r-1)$*-dimensional manifold* V_{r-1}^{r-1} *of order* $r-1$.

To prove this, we note that for $r=2$ there is one line joining two given points in a plane and that for $r=3$ the lines meeting three given lines in an S_3 form a quadric surface. If $r=4$, that is, if four planes are given in S_4, the locus of the ∞^2 lines incident with them is a V_3^3. This cubic hypersurface with its many interesting properties has been studied by a number of writers.*

If $r=5$, that is, if five three-spaces are given in S_5, pass an S_4 through one of them, say R_3. This S_4 meets the other four 3-spaces in four planes and the lines incident with these four planes are also incident with R_3. These lines form a V_3^3 in S_4. The manifold of the ∞^3 lines incident with the five given 3-spaces is intersected by the S_4 through R_3 in R_3 and a V_3^3 and is therefore of order 4. If we apply this process of reasoning to the cases $r=6$, 7, etc., we soon arrive at the general theorem stated above.

Consider another proof. Let the r given $(r-2)$-spaces in S_r be S'_{r-2}, S''_{r-2}, $\cdots$, $S^{(r)}_{r-2}$, and further let a general line l be given. The points of l determine with S'_{r-2}, S''_{r-2}, $\cdots$, $S^{(r-1)}_{r-2}$ $r-1$ projective pencils of hyperplanes. The rth $(r-2)$-space, $S^{(r)}_{r-2}$, intersects these pencils in $r-1$ pencils of $(r-3)$-spaces. As there are $r-1$ sets of corresponding $(r-3)$-spaces each intersecting in a point, there are $r-1$ lines of intersection of corresponding hyperplanes of the pencils in S_r which meet $S^{(r)}_{r-2}$. Hence, the general line l meets $r-1$ of the ∞^{r-2} lines

* See Bertini, *Projecktive Geometrie Mehrdimensionaler Räume*, 1924, Chap. 8, §§25–36, where references are given.

incident with the r given $S_{r-2}^{(i)}[i=1, 2, \cdots, r]$, and therefore the manifold of the ∞^{r-2} lines is of order $r-1$.

It is of interest to note that the V_{r-1}^{r-1} in question contains the r given $(r-2)$-spaces $S_{r-2}^{(i)}$ $[i=1, 2, \cdots, r]$. These $S_{r-2}^{(i)}$ intersect ν by ν in $\binom{r}{\nu}$ $(r-2\nu)$-spaces $S_{r-2\nu}^{(j)}$ $[\nu=1, 2, \cdots, r/2$ if r is even, $\nu=1, 2, \cdots, (r-1)/2$ if r is odd; $j=1, 2, \cdots, \binom{r}{\nu}]$. The V_{r-1}^{r-2} contains all these $S_{r-2\nu}^{(j)}$ ν-ply. Any V_{r-1}^{r-1} in S_r containing r $(r-2)$-spaces is a hypersurface of this type, for any line meeting these r $(r-2)$-spaces lies entirely on the hypersurface and there are ∞^{r-2} such. For the case $r=4$, there are additional planes and conical points. The V_3^3, besides containing the four given planes and the six conical points in which they intersect two by two, contains eleven other planes and four other conical points. These fifteen planes and ten nodes are such that each plane contains four nodes and each node is on six planes.

The equation of the V_{r-1}^{r-1} we are considering is not yet known. The writer has derived the following equation for V_3^3 in S_4:

$$x_0x_1x_4 + x_1x_2x_3 + x_2x_3x_4 - x_0x_2x_3 - x_1x_2x_4 - x_1x_3x_4 = 0.$$

The ten conical points are the vertices of the coordinate simplex, the unit point and the following:

(1:1:1:0:0), (1:0:1:0:1), (1:0:0:1:1), (1:1:0:1:0).

By the linear transformation

$$\begin{aligned}
\rho x_0 &= - y_2 + y_3,\\
\rho x_1 &= y_0 + y_1 + y_3 + y_4,\\
\rho x_2 &= y_1 + y_3,\\
\rho x_3 &= y_0 + y_3,\\
\rho x_4 &= - y_2 - y_4
\end{aligned}$$

the above equation is transformed into

$$y_0^3 + y_1^3 + y_2^3 + y_3^3 + y_4^3 + y_5^3 = 0,$$

where

$$y_0 + y_1 + y_2 + y_3 + y_4 + y_5 = 0.$$

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