HOMOGRAPHIC CIRCLES OR CLOCKS*

BY LULU HOFMANN AND EDWARD KASNER

1. *Introduction.* This paper contains some theorems on the homographic transformation of one circle into another. The distribution of points on the transformed circle is studied under the assumption that the distribution on the original circle is uniform. The varying *density* on the transformed circle is characterised *im Grossen* by the *centroid,* a point defined analytically by a mean-value process. At an individual point it is measured by the absolute value of the ratio of corresponding arc elements on the two circles, the ratio taken from the original to the transformed element.

The distribution of points on the transformed circle proves to be uniform when and only when the centroid coincides with the center. When this coincidence does not occur, the density varies as follows.

It reaches its maximum and its minimum, two reciprocal values, in the end points of the diameter through the centroid, the *main diameter.*

It assumes every intermediate value twice and to every value occurring the reciprocal value likewise occurs.

Points of equal density lie symmetrically with regard to the main diameter, and points of reciprocal densities are collinear with the centroid.

If we draw a chord through the centroid and number the end points and the corresponding segments determined by the centroid arbitrarily as the first and second, then the density in the first point will be equal to the ratio of the length of the second segment to that of the first segment.

2. *Homographic Clocks. The Centroid.* We shall treat the problem analytically. As the most convenient tool we choose the linear fractional transformation with complex coefficients of one complex variable into another

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$$z = \frac{A + Bt}{C + Dt} \equiv \Omega(t).$$

This transforms the unit circle T,

$$t = e^{i\theta}, \qquad (\theta \text{ real}),$$

into the circle Z

$$z = \frac{A + Be^{i\theta}}{C + De^{i\theta}} \equiv w(\theta),$$

and these are the two circles we shall study.* They are homographically related on account of the linearity of the transformation formula. Geometrically the transformation is determined by three pairs of corresponding points on the two circles, because, since we are dealing with a homography, the anharmonic ratio of four points on T is equal to that of the four corresponding points on Z. A circle whose points are in definite homographic correspondence with the points of the unit circle, distinguished points we term a *homographic clock*; and more precisely a *positive* or *negative homographic clock* according as when t moves on the circle T in a certain sense, z moves on the circle Z in the same or in the opposite sense (the sense is uniquely determined on account of the one-to-one character of the transformation).

The concept of *homographic clock* belongs to both *inversive* and *projective* geometry.

With regard to the unit circle T, we note that $t = -C/D$ and $t = -\bar{D}/\bar{C}$ are corresponding points in inversion. The same relation must therefore hold with regard to the circle Z for the transformed points

$$z = \Omega\left(-\frac{C}{D}\right) = \infty, \quad z = \Omega\left(-\frac{\bar{D}}{\bar{C}}\right) = \frac{A\bar{C} - B\bar{D}}{C\bar{C} - D\bar{D}}.$$

And since the point at infinity is the conjugate of the center of the circle of inversion, it follows that Z has the center

$$N = \frac{A\bar{C} - B\bar{D}}{C\bar{C} - D\bar{D}}.$$

*When in the following z occurs and nothing is expressly remarked to the contrary, then $z = w(\theta)$ is always meant.

Furthermore, forming $|N-w(0)|$ (we could also have chosen any other definite value for θ) we find that Z has the radius,

$$R = \left|\frac{AD - BC}{C\bar{C} - D\bar{D}}\right| .$$

The transformed circle Z will therefore degenerate into a line when and only when $|C| = |D|$, which we henceforth exclude. To obtain a general idea of the distribution of points on the new circle, we form the mean value M of z with regard to θ

$$M = \frac{1}{2\pi}\int_0^{2\pi}\frac{A + Be^{i\theta}}{C + De^{i\theta}}\,d\theta .$$

The point M indicates the *mean point* into which the points of T are transformed. We call it the *centroid* of the clock. For $D=0$ and $C=0$ the formulas simplify; we have

$$D = 0:\quad z = \frac{A}{C} + \frac{B}{C}e^{i\theta},\quad N = \frac{A}{C},\quad M = \frac{A}{C};$$

$$C = 0:\quad z = \frac{A}{D}e^{-i\theta} + \frac{B}{D},\quad N = \frac{B}{D},\quad M = \frac{B}{D} .$$

When neither $D=0$ nor $C=0$, the integrand becomes

$$\frac{B}{D} + \frac{AD - BC}{D}\cdot\frac{1}{C + De^{i\theta}},$$

and

$$M = \frac{1}{2\pi}\left[\frac{B}{D}\theta + \frac{1}{i}\left(\frac{A}{C} - \frac{B}{D}\right)\left(i\theta - \log\{C + De^{i\theta}\}\right)\right]_0^{2\pi}$$

$$= \frac{A}{C} - \frac{1}{2\pi i}\left(\frac{A}{C} - \frac{B}{D}\right)\epsilon,$$

where $\epsilon = 0$ when $|C| > |D|$ and $\epsilon = 2\pi i$ when $|C| < |D|$. Including the cases $D=0$ and $C=0$, we therefore have the following result.

The centroid M of the clock Z lies at the point

$$M = \frac{A}{C}, \quad |C| > |D|,$$

$$M = \frac{B}{D}, \quad |C| < |D|.$$

As the formulas show, the center and the centroid coincide when $D=0$ or $C=0$. Vice versa, these are the only cases for which this coincidence occurs. For according to the two values of M we have

$$N - M = \frac{A\bar{C} - B\bar{D}}{C\bar{C} - D\bar{D}} - \frac{A}{C} = \frac{\bar{D}(AD - BC)}{C(C\bar{C} - D\bar{D})},$$

$$N - M = \frac{A\bar{C} - B\bar{D}}{C\bar{C} - D\bar{D}} - \frac{B}{D} = \frac{\bar{C}(AD - BC)}{D(C\bar{C} - D\bar{D})},$$

and these expressions vanish when and only when $D=0$ or $C=0$, because $AD-BC=0$ would make z a constant independent of θ.

Among all homographic clocks, those of the types $z=A+Be^{i\theta}$ and $z=Ae^{-i\theta}+B$ corresponding to $D=0$ and $C=0$ are thus distinguished by the coincidence of N and M. It follows immediately from the formulas that when the point t moves with uniform velocity on the circle T, the point z moves with uniform velocity on the circle Z, in the same sense as t in the first case and in the opposite sense in the second case. We therefore term these clocks *uniform positive* and *negative clocks.*

The uniform positive and negative clocks

$$z = A + Be^{i\theta}, \quad z = Ae^{-i\theta} + B$$

are the only homographic clocks for which the center and the centroid coincide.

From the preceding discussion, by reason of continuity, we may deduce the following theorem.

Any homographic clock

$$z = \frac{A + Be^{i\theta}}{C + De^{i\theta}}$$

is a positive or negative clock according as $|C| > |D|$ *or* $|C| < |D|$.

3. *The Density.* After having thus studied the distribution of points on Z *im Grossen* by constructing the mean point or centroid M, we now proceed to study it in the infinitely small, that is, at the individual points of Z. It will be represented by a quantity which for obvious reasons we call the *density* and abbreviate by δ. The density will be characterised by $|dz/d\theta|$, the ratio of the corresponding arc lengths on the circles Z and T. For example, let us consider the circle T as uniformly covered with mass in such a manner that the mass on $d\theta$ is measured by $|d\theta|$. Then, since under the transformation which transforms $d\theta$ into dz, the mass may be considered constant, the density δ with which it is distributed on dz must be such that $|dz| \cdot \delta = |d\theta|$, that is, the density will be the reciprocal of $|dz/d\theta|$.

The formulas simplify, if we do not study $|dz/d\theta|$ itself but the related expression $|dz'/d\theta|$, where

$$z' = (z - N)\frac{D\bar{D} - C\bar{C}}{AD - BC} = \frac{\bar{D} + \bar{C}e^{i\theta}}{C + De^{i\theta}}\cdot$$

The circle Z' is obtained from Z by three simple operations in the z-plane: the translation represented by the vector $-N$ which carries the center of Z into the origin; the rotation through the argument of $(D\bar{D} - C\bar{C})/(AD - BC)$, and the magnification from the origin with the factor $1/R$. Hence it is obvious that the distribution of points on Z' is essentially the same as on Z (namely equivalent by similitude). We therefore omit the primes and continue with the circle

$$z = \frac{\bar{D} + \bar{C}e^{i\theta}}{C + De^{i\theta}} \equiv w(\theta)$$

which we now call Z. It is a circle of unit radius with its center at the origin. To fix on a definite case, let us assume $|C|>|D|$, so that

$$M = \frac{\bar{D}}{C}.$$

It will be seen immediately from the results obtained that they hold equally for the other case.

By differentiation we obtain

$$\left|\frac{dz}{d\theta}\right| = \left|\frac{ie^{i\theta}(C\bar{C} - D\bar{D})}{(C + De^{i\theta})^2}\right| = \frac{C\bar{C} - D\bar{D}}{C\bar{C} + D\bar{D} + \bar{C}De^{i\theta} + C\bar{D}e^{-i\theta}}.$$

Putting

$$|C| = c, \quad |D| = d, \quad \bar{C}D = cde^{ik},$$

we obtain

$$\left|\frac{dz}{d\theta}\right| = \frac{c^2 - d^2}{c^2 + d^2 + 2cd\cos(\theta + k)} \equiv \frac{1}{\delta(\theta)}.$$

The density on the transformed circle Z is independent of θ when and only when $D=0$ or $C=0$, that is when the center and the centroid coincide. This means that the clock is uniform.

By substituting $\theta' = \theta + k$ we obtain

$$z = \frac{\bar{C}d}{Dc}\cdot\frac{d + ce^{i\theta'}}{c + de^{i\theta'}} = w'(\theta'), \quad \left|\frac{dz}{d\theta}\right| = \frac{c^2 - d^2}{c^2 + d^2 + 2cd\cos\theta'} \equiv \frac{1}{\delta'(\theta')}.$$

It is obvious that the points $z = w'(\theta')$ and $z = w'(-\theta')$ lie symmetrically with regard to the diameter through the point $z = \bar{C}d/(Dc)$; and since $\delta'(\theta') = \delta'(-\theta')$, the density at these points is the same. (Vice versa any two points of equal density have the above described mutual position.) At the point $z = \bar{C}d/(Dc)$ itself, $\theta' = 0$, and δ reaches its maximum

$$\delta_{max} = \frac{c + d}{c - d}.$$

At the diametrically opposed point $z = -\bar{C}d/(Dc)$, $\theta' = \pi$ and δ assumes its minimum

$$\delta_{min} = \frac{c - d}{c + d} = \frac{1}{\delta_{max}}.$$

Referring to the maximum and minimum value of the density, we denote these points by z_{max} and z_{min}:

$$z_{max} = \frac{\overline{C}d}{Dc}, \quad z_{min} = -\frac{\overline{C}d}{Dc}.$$

The diameter through z_{max} and z_{min} we term *the main diameter.*

The distribution of points on the circle Z is symmetric with regard to a definite diameter, the main diameter. The density is equal in points lying symmetrically with regard to this diameter and reaches its maximum and minimum, which are reciprocal, in its end points.

We have

$$\frac{z_{max}}{M} = \frac{c}{d}, \quad \frac{z_{min}}{M} = -\frac{c}{d},$$

$$\left| \frac{z_{max} - M}{z_{min} - M} \right| = \frac{c - d}{c + d} = \delta_{min} = \frac{1}{\delta_{max}}.$$

This means:

The centroid lies on the main diameter and divides it in such a manner that the ratio of its distance from the first end point to its distance from the second end point is equal to the density at the second point.

It is obvious, from the fact that the maximum and minimum density are reciprocal, that to every value of the density occurring the reciprocal value must also occur. We shall now show that for points with reciprocal densities a theorem holds which is the immediate extension of the one just stated.

Denote by $z = w(\theta)$ and $z^* = w(\theta^*)$ any two points collinear with the centroid and by l and l^* their respective distances from this point. We have

$$l = \left| \frac{\bar{D} + \bar{C}e^{i\theta}}{C + De^{i\theta}} - \frac{\bar{D}}{C} \right| = \left| \frac{(c^2 - d^2)e^{i\theta}}{C(C + De^{i\theta})} \right|.$$

Without calculating θ^* itself, one easily finds

$$l^* = \left| \frac{c^2 + d^2 + 2cd\cos(\theta + k)}{C(C + De^{i\theta})} \right|,$$

so that

$$\frac{l}{l^*} = \frac{c^2 - d^2}{c^2 + d^2 + 2cd\cos(\theta + k)} = \frac{1}{\delta(\theta)}.$$

But similarly of course

$$\frac{l^*}{l} = \frac{1}{\delta(\theta^*)}.$$

Therefore

$$\delta(\theta) = \frac{1}{\delta(\theta^*)}.$$

On the transformed circle Z the density is reciprocal in two points z and z^* which are collinear with the centroid. The density at the point $z(z^*)$ is equal to the ratio of the distances of $z^*(z)$ and $z(z^*)$ from the centroid.

Appendix on Polygenic Functions

BY EDWARD KASNER

The geometric concept of a *clock* first presented itself in the theory of *polygenic functions*†

$$w = \phi(x,y) + i\psi(x,y),$$

as follows. The derivative

$$\frac{dw}{dz} = L + le^{-2i\theta},$$

where

$$L = \mathfrak{D}w = \tfrac{1}{2}(D_x - iD_y)w,$$

$$l = \mathfrak{P}w = \tfrac{1}{2}(D_x + iD_y)w.$$

† See E. Kasner, Science, vol. 66 (1927), pp. 581–582, and Proceedings of the National Academy of Sciences, vol. 14 (1928), pp. 75–82.

This gives for each point $x+iy$ a uniform negative clock with center vector L and phase vector l.

It is to be noticed that the angle θ of the preceding joint paper is here 2θ because we are dealing with a correspondence between the points of the derivative circle and a pencil of directions, rather than with a unit circle.*

If w_1 and w_2 are any two polygenic functions, then the derivative is

$$\frac{dw_1}{dw_2} = \frac{L_1 + l_1 e^{-2i\theta}}{L_2 + l_2 e^{-2i\theta}},$$

and so defines a general homographic clock. In particular, the clock degenerates into a point when and only when the two functions are of the same class.† The clock becomes a straight line (the case omitted in the joint paper) when and only when the jacobian of ϕ_2 and ψ_2 vanishes. Finally, the clock becomes uniform (negative or positive) when and only when the denominator function w_2 is an analytic function of $x+iy$ or $x-iy$. Hence the mean value (centroid)

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{dw_1}{dw_2} d\theta$$

coincides with the center of the derivative clock when and only when w_2 is an analytic function of $x+iy$ or $x-iy$. The theory of congruences of clocks will be studied elsewhere.

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* See the abstracts in this Bulletin, vol. 34 (1928), 152, pp. 263–264.

† According to the terminology introduced by Hedrick, Ingold, and Westfall, *Theory of non-analytic functions of a complex variable*, Journal de Mathématiques, (9), vol. 2 (1923), 327–342.