THE POLAR CURVES OF PLANE ALGEBRAIC CURVES IN THE GALOIS FIELDS*

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By imitating the proofs in Fine's *College Algebra* (pp. 460–462) and Veblen and Young's *Projective Geometry* (vol. I, pp. 255–256) we can readily show that also in the Galois fields of order p^n (p a prime integer) we have Taylor's expansion

$$f(x+\lambda X,\, y+\lambda Y,\, z+\lambda Z)$$
$$\equiv f(x,\, y,\, z)+\frac{\lambda}{1!}(f_x' X+f_y' Y+f_z' Z)$$
$$+\frac{\lambda^2}{2!}(f_x' X+f_y' Y+f_z' Z)^{(2)}+\cdots$$
$$+\frac{\lambda^r}{r!}(f_x' X+f_y' Y+f_z' Z)^{(r)}+\cdots+f(X,\, Y,\, Z)=0,$$

where $(f_x' X+f_y' Y+f_z' Z)^{(i)}$ is symbolic for an expression containing derivatives of the ith order, and $f(x, y, z)=0$ is an algebraic curve of order n. In the above expansion we must take all the derivatives as though p were not a modulus, cancel out common factors from numerators and denominators, and then set $p=0$.

The rth polar of (X, Y, Z) with respect to $f(x, y, z)=0$ is

$$\frac{1}{r!}(f_x' X+f_y' Y+f_z' Z)^{(r)}=0.$$

In particular the rth polar of $(1, 0, 0)$ is $(1/r!)\partial^r f(x, y, z)/\partial x^r=0$. We suppose first of all that n has the value

$$n=\alpha p^m+\beta p^{m-1}+\cdots+\gamma p^2+\delta p+\epsilon,$$
$$\epsilon\neq 0,\quad p=\epsilon+\zeta,\quad \zeta\neq 0.$$

* Presented to the Society, December 28, 1927.

We can write the polars of (1, 0, 0) by a sort of detached coefficients, underlining the coefficients that have p as a factor, as follows:

$$(1/1!)[n, n-1, n-2, \cdots, \underline{n-\epsilon}, n-\epsilon-1,$$
$$\cdots, \underline{n-p-\epsilon}, \cdots, \underline{n-p^2-\epsilon}, \cdots, 3,2,1] = 0,$$

$$(1/2!)[n(n-1), (n-1)(n-2), \cdots, (n-\epsilon+1)(\underline{n-\epsilon}),$$
$$(\underline{n-\epsilon})(n-\epsilon-1), \cdots, (n-p-\epsilon+1)(\underline{n-p-\epsilon}),$$
$$(\underline{n-p-\epsilon})(n-p-\epsilon-1), \cdots, (n-p^2-\epsilon+1)$$
$$\cdot(\underline{n-p^2-\epsilon}), (\underline{n-p^2-\epsilon})(n-p^2-\epsilon-1), \cdots, 3\cdot 2, 2\cdot 1] = 0,$$

. . . .

$$[1/(\epsilon+1)!][n(n-1)(n-2)\cdots(\underline{n-\epsilon}), (n-1)(n-2)\cdots$$
$$\cdot(\underline{n-\epsilon})(n-\epsilon-1), \cdots, (\underline{n-\epsilon})(n-\epsilon-1)\cdots(n-2\epsilon),$$
$$(n-\epsilon-1)(n-\epsilon-2)\cdots(n-2\epsilon)(n-2\epsilon-1), \cdots,$$
$$(\epsilon+1)!] = 0,$$

. . . .

$$(1/p!)[n(n-1)\cdots(\underline{n-\epsilon})\cdots(n-p+1)\cdots, p!] = 0,$$

. . . .

where $(n-\lambda)$ $(n-\lambda-1)\cdots(n-\lambda-i)$ stands for all the terms of the same $(n-\lambda-i-1)$ power, which then have this common factor in their coefficients. From the above polars we see that the ϵth polar has at (1, 0, 0) a tangent having $(\epsilon+1)$-point contact if (1, 0, 0) is not on $f(x, y, z)=0$, otherwise a multiple point of order $\epsilon+1$. The $(\epsilon+1)$th polar, $(\epsilon+2)$th, $\cdots$, $(p-1)$th polar all have multiple points of order $\epsilon+1$ at (1, 0, 0). Similarly the $(p+\epsilon+1)$th polar, $(p+\epsilon+2)$th, $\cdots$, $(2p-1)$th have at (1, 0, 0) multiple points of order $\epsilon+1$; also the $(2p+\epsilon+1)$th polar points of order $(3p-1)$ h, $\cdots$, the $(\theta p^i+\cdots+\phi p+\epsilon+1)$th polar points of order $(\theta p^i+\cdots+\phi p+p-1)$th, etc. Moreover we note that if any one of the polar curves that have multiple points at (1, 0, 0) is a curve of degree $\epsilon+1$, then this polar curve is degenerate. Thus for $p=2$, $n=2^2+1$, $\epsilon=1$, we find the 2d polar is degenerate; for $p=3$, $n=3+1$, $\epsilon=1$, we find again the 2d polar is degenerate.

If $n=\alpha p^m+\beta p^{m-1}+\cdots+\gamma p^2+\delta p$, i.e. $\epsilon=0$ in n, then all the polars of (1, 0, 0) pass through (1, 0, 0) whether or not this point lies on $f(x, y, z)=0$.

If $n<p$ we find no peculiarities like the above.

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THE CHARACTERISTIC EQUATION OF A MATRIX*

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1. *Introduction.* Consider any square matrix A, real or complex, of order n. If I is the unit matrix, $A-\lambda I$ is called the *characteristic matrix* of A; the determinant of the characteristic matrix is called the *characteristic determinant* of A; the equation obtained by equating this determinant to zero is called the *characteristic equation* of A; and the roots of this equation are called the *characteristic roots* of A. If A happens to be a matrix of a particular type certain definite statements may be made as to the nature of its characteristic roots. For example, if A is Hermitian its characteristic roots are all real; if A is real and skew-symmetric, its characteristic roots are all pure imaginary or zero; if A is a real orthogonal matrix, its characteristic roots are of modulus unity. However, if A is not a matrix of some special type, no general statement can be made as to the nature of its characteristic roots. In 1900 Bendixson† proved that if $\alpha+i\beta$ is a characteristic root of a real matrix A, and if $\rho_1\geqq\rho_2\geqq\cdots\geqq\rho_n$ are the characteristic roots (all real) of the symmetric matrix $\frac{1}{2}(A+A')$, then $\rho_1\geqq\alpha\geqq\rho_n$. The extension to the case where the elements of A are com-

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* Presented to the Society, December 28, 1927.

† Bendixson, *Sur les racines d'une équation fondamentale*, Acta Mathematica, vol. 25 (1902), pp. 359–365.