AN ANALYSIS OF SOME GENERAL PROPOSITIONS*

BY C. H. LANGFORD

1. *Introduction.* General propositions are commonly described as being those propositions which arise from matrices by generalization, that is, as being such propositions as can be derived from some matrix by the attachment of an applicative, "some" or "every," to each variable constituent of the matrix.† In this paper an analysis of general propositions is suggested, which results from a slightly different view of the relations of general propositions to matrices. It appears that the analysis which is suggested involves a generalization of the ordinary analysis.

2. *Unanalysed Propositions.* We may begin with a consideration of elementary matrices whose values are elementary functions of unanalysed propositions, such as, for example, $p \supset q$. Let t_0 denote any elementary proposition. Then we can write, for example, $(t_0).t_0 \vee \sim t_0$, that is, every elementary proposition is true or false. If p_0, q_0 denote elementary propositions, then, of course, $p_0 \supset q_0$ denotes elementary propositions; but this latter, more complex function denotes a narrower range of propositions than does t_0, since whatever is an elementary proposition of the form $p_0 \supset q_0$ must be an elementary proposition of the form t_0, but not conversely. Let t_1 denote any elementary matrix. Then t_1 denotes what denotes elementary propositions; t_1 denotes t_0, $p_0 \supset q_0$, and the like, which denote elementary propositions. Now we can form such functions as $p_1 \supset q_1$, which denote a narrower range of matrices than does t_1.

Since elementary matrices are neither true nor false, we cannot say $(t_1).t_1 \vee \sim t_1$; but we can say

$$(t_1):(t_0).t_0 \vee \sim t_0,$$

---

* Presented to the Society, September 9, 1927.

† See *Principia Mathematica*, second edition, vol. 1, p. xxiii.

that is, every value of every matrix is true or false. Variables such as t_0, t_1 fall into a hierarchy of values and values of values. We can write such propositions as $(t_1):(\exists t_0).t_0$, that is, some value of every matrix is true (which is false), and $(\exists t_1):(t_0).t_0$, that is, all values of some matrix are true (which is true); and we have, of course,

$$(t_1)(t_0).t_0.\vee.(\exists t_1)(\exists t_0).\sim t_0,$$
$$(t_1)(\exists t_0).t_0.\vee.(\exists t_1)(t_0).\sim t_0,$$

and the like. Consider the proposition

$$(t_1).(t_0)t_0 \vee (\exists t_0)\sim t_0,$$

that is, every matrix is such that all of its values are true or at least one of its values is false. This proposition, although a consequence of the proposition $(p).p \vee \sim p$, cannot be stated without the use of some variable such as t_1. We have also, of course,

$$(t_2):(t_1)(\exists t_0).t_0.\vee.(\exists t_1)(t_0).\sim t_0,$$

which is different from

$$(t_1)(\exists t_0).t_0.\vee.(\exists t_1)(t_0).\sim t_0\ ;$$

and it would seem that the proposition

$$(\exists t_2):(t_1)(\exists t_0).t_0.\vee.(\exists t_1)(t_0).\sim t_0$$

is not equivalent to any proposition which can be stated without the use of some variable such as t_2.

3. *Values of Functions and Species of Functions.* There are two ways in which inferences can be drawn from universal propositions of the kind with which we are concerned: we can replace a universally quantified function by one of the values which it denotes, and we can replace a universally quantified function, as genus, by a more determinate function, as species.* Thus,

$$(t_1).(\exists t_0)t_0 \vee (t_0)\sim t_0$$

* See *Principia Mathematica*, second edition, vol. I, p. xxiii.

implies

$$\exists(p_0 \vee q_0) . p_0 \vee q_0 . \vee . (p_0 \vee q_0) . \sim p_0 . \sim q_0,$$

that is,

$$(\exists p_0, q_0) . p_0 \vee q_0 . \vee . (p_0, q_0) . \sim p_0 . \sim q_0 .$$

Again,

$$(t_1)(\exists t_0) . t_0 \vee \sim t_0$$

implies

$$(p_1 \vee q_1) \exists (p_0 \vee q_0) : p_0 \vee q_0 . \supset . p_0 \vee q_0,$$

that is,

$$(p_1, q_1)(\exists p_0, q_0) : p_0 \vee q_0 . \supset . p_0 \vee q_0 .$$

It is to be noted that *genus* and *species* are to be understood in intension: being a function of the form $p_1 \vee q_1$ entails being a function of the form t_1; and this is the ground of the implication. There are two ways in which particular propositions can be inferred: we can infer a particular proposition from an instance of the function which is quantified particularly, and we can infer a particular proposition from a more determinate particular proposition. Thus, $(t_0) . t_0 \vee \sim t_0$ implies $(\exists t_1)(t_0) . t_0$; and $\exists(p_0 . q_0) . p_0 . q_0$ implies $(\exists t_0) . t_0$.

4. *Analysed Propositions.* Heretofore we have been concerned with matrices whose values are elementary functions of unanalysed propositions, and we have illustrated the way in which a hierarchy of functions can be formed in this connection. Thus we may have

$$p_2 \vee q_2,$$

which has as a typical value

$$p_1' . p_1'' . \vee . q_1' . q_1'',$$

which has as a typical value

$$r_0' \supset s_0' . r_0'' \supset s_0'' . \vee . t_0' \supset u_0' . t_0'' \supset u_0'',$$

which has as a value any elementary proposition of this form.* I wish to explain how a similar hierarchy can be formed in connection with analysed propositions. A propo-

* See *Principia Mathematica*, loc. cit., p. xxxi, etc.

sition is a value of a function if the function can be obtained by replacing constituents of the proposition by appropriate variables. We are to consider propositions which are values of relational functions, such as the proposition "*a* gives *b* to *c*." This proposition is a value of each of the functions "*a* gives *b* to *x*," $f(a, b, c)$, $f(a)$, $f(x)$, $f(x, y, z)$, among others. Commonly, one and the same proposition is open to various analyses. If we consider an appropriate constituent of a relational proposition, two entities appear which can be replaced by variables, namely this constituent and the remainder of the proposition. It is, however, often difficult to decide what are the constituents of a proposition which can be replaced by variables.

Now a proposition of the form *Rab* is a value of the function *Ray*; and it is clear that *Ray* can be obtained from *Rxy* by replacing *x* by *a*. We may indicate this order of denoting by writing Rx_1y_0; so that Rx_1y_0 denotes Ray_0, which denotes *Rab*. Subscripts indicate parameters of various orders, and accordingly they indicate the order of denoting; so that the subscript which attaches to a variable determines the point in the hierarchy at which the variable takes values. Thus a function $f(x, y, z)$ is ambiguous as regards the parametric order of its variable constituents. We may have $f_1(x_0, y_0, z_0)$, which denotes "x_0 gives y_0 to z_0," which denotes "*a* gives *b* to *c*"; or, we may have $f_0(x_1, y_1, z_1)$, which denotes $f_0(a, b, c)$, which denotes "*a* gives *b* to *c*"; and so on. It is clear that, as a result of this analysis, we can attach an applicative, "some" or "every," to the entire function at each stage of the hierarchy. If we apply "some" to $f_1(x_0, y_0, z_0)$ and "every" to $f_0(x_0, y_0, z_0)$, we have

$$\exists f_1(x_0, y_0, z_0) : (f_0(x_0, y_0, z_0)) . f_0(x_0, y_0, z_0),$$

which corresponds to $(\exists f) : (x, y, z) . f(x, y, z)$ in the ordinary analysis; whereas, if we apply "some" to $f_0(x_1, y_1, z_1)$ and "every" to $f_0(x_0, y_0, z_0)$, we have

$$\exists f_0(x_1, y_1, z_1) : (f_0(x_0, y_0, z_0)) . f_0(x_0, y_0, z_0),$$

which corresponds to $(\exists x, y, z) : (f) . f(x, y, z)$.

Now although the proposition $(f_1x_0):(f_0x_0).f_0x_0$ corresponds to $(f):(x).fx$, there is an important difference between these propositions which I wish to point out. When we say $(f):(x).fx$, it is presupposed that each value of x can be combined with every value of f to form a significant proposition; whereas, no such presupposition as this is involved when we write $(f_1x_0):(f_0x_0).f_0x_0$. If there should be values of f_0, say f', f'', and values of x_0, say x', x'', such that $f'x'$ and $f''x''$, but such that $f'x''$ and $f''x'$ are non-significant, this fact would not render $(f_1x_0):(f_0x_0).f_0x_0$ either false or non-significant. A matrix is to be regarded as taking such values as exhibit the form of the matrix; and the values taken by a variable constituent of a matrix depend upon the matrix in which the variable occurs. It is therefore possible that $(f_1x_0):(f_0x_0).f_0x_0$ should have a wider reference than $(f):(x).fx$.

5. *Analysed and Unanalysed Propositions.* I wish now to consider functions whose values are elementary functions of analysed propositions, and to inquire how, if at all, the denotative hierarchy for elementary functions of unanalysed propositions can be combined with the hierarchy for analysed propositions. We have such propositions as $(f_1x_0)(\exists f_0x_0).f_0x_0$ and $(\exists f_0x_1)(f_0x_0).f_0x_0$. Now it appears that we can extend the range of t_0, t_1, $\cdots$ so as to include such functions as f_0x_0, f_1x_0, $\cdots$. This can be seen to be possible by noting that f_0x_0 is a species of t_0, and that f_1x_0 is a species of t_1. An analysed function will have the order of its constituents of highest parametric order. It is clear that, for example, $(f_0x_0).f_0x_0 \vee \sim f_0x_0$ follows from $(t_0).t_0 \vee \sim t_0$, and that $(f_0x_0)f_0x_0 \vee (\exists f_0x_0)\sim f_0x_0$ follows from $(t_0).t_0 \vee (\exists t_0)\sim t_0$; and also that $(\exists f_0x_0)f_0x_0$ entails $(\exists t_0)t_0$. Consider the proposition

$$(a) \qquad (t_1).(\exists t_0)t_0 \vee (t_0)\sim t_0.$$

If we choose f_1x_0 as a species of t_1, (a) is seen to entail

$$(f).(\exists x)fx \vee (x)\sim fx\ ;$$

but if we choose $f_0 x_1$ as a species of t_1, (a) is seen to entail

$$(x) . (\exists f) fx \vee (f) \sim fx .$$

Functions of any number of variables can replace t_1, but, in any case, doubly-quantified propositions result. Of course, we can assign values to t_1, instead of replacing it by species. We can, for example, substitute fx_0 (where f is constant), so that we have

$$(\exists x) fx \vee (x) \sim fx ;$$

or, if we substitute $f_0 a$, we have

$$(\exists f) fa \vee (f) \sim fa .$$

This analysis gives us a certain advantage as against the ordinary analysis. In the ordinary analysis, we can write $(p) . p \vee \sim p$, and derive, as a species, $(f, x) . fx \vee \sim fx$; but this is confined to singly-quantified propositions, that is, to propositions involving a single applicative, "some" or "every"; whereas, we are now able to write, say, $(p_n) \cdots (\exists p_0) . p_0 \vee \sim p_0$, which is a proposition involving $n+1$ applicatives.

6. *Multiply-Quantified Propositions.* Let $F(t_0)$ denote an elementary function of t_0. Then it is clear that $(t_0) . F(t_0)$ implies $(t_1) : (t_0) . F(t_0)$. Similarly, $(t_1) : (t_0) . F(t_0)$ implies $(t_2) :. (t_1) : (t_0) . F(t_0)$. Now $(\exists t_0) . F(t_0)$ can be obtained from $(t_0) . F(t_0)$; and generally, in a function of the form $(t_n) \cdots F(t_0)$, we can turn any universal variable into a particular variable. Accordingly, any proposition on $F(t_0)$, of whatever degree of quantification, can be obtained from $(t_0) . F(t_0)$.

In the argument of the last paragraph, we have used $F(t_0)$ to denote an elementary function of t_0; but I should like to point out that it is not at all necessary to employ such expressions as $F(t_0)$. The proposition $(\exists F) : (t_0) . F(t_0)$, for example, expresses what is expressed by $(\exists t_1) : (t_0) . t_0$. We can express the proposition

$$(F) : (t_0) . F(t_0) . \supset . (t_1)(t_0) . F(t_0)$$

in the following way. Note that $(\exists t_0) . t_0$ is equivalent to $(\exists t_1)(\exists t_0) . t_0$; so that we can write

$$(t_2) : (\exists t_1)(\exists t_0) . \sim t_0 . \vee . (t_1)(t_0) . t_0 .$$

Mr. Russell has shown how elementary functions of propositions that are not elementary can be derived from elementary matrices.* Thus, $(x) . fx \vee \sim fx$ implies $(x)(\exists y) . fx \vee \sim fy$, which is equivalent to $(x) fx \vee (\exists y) \sim fy$. In precisely the same way,

$$(t_1)(t_0) . t_0 \vee \sim t_0$$

implies

$$(t_1) : (t_0)(\exists t_0') . t_0 \vee \sim t_0' ,$$

which is equivalent to $(t_1) . (t_0) t_0 \vee (\exists t_0) \sim t_0$, which implies

$$(t_1)(\exists u_1) . (u_0) u_0 \vee (\exists t_0) \sim t_0 ,$$

which is equivalent to

$$(\exists t_1)(t_0) t_0 \vee (t_1)(\exists t_0) \sim t_0 .$$

It is clear that elementary functions of propositions of the first and second orders can be derived in this way, whatever the degree of quantification of these propositions. Moreover, from $(t_1) . (\exists t_0) t_0$ we derive both $(\exists x) fx$ and $(\exists f) fa$, although one of these latter propositions would be said to be of the first order and the other of the second order. Furthermore, in a proposition such as $(t_0) t_0$ there seems to be no reason for restricting t_0 to elementary propositions; it can denote propositions of the first and second orders as well. Let "$(\exists x) . a$ gives b to x" be denoted by $f(a, b)$. Then

$$(\exists a, b) f(a, b) \vee (a, b) \sim f(a, b)$$

results from

$$(t_1) . (\exists t_0) t_0 \vee (t_0) \sim t_0 .$$

It does not matter that f is not elementary, since it occurs as an unanalysed constituent.

HARVARD UNIVERSITY

---

* See *Principia Mathematica*, loc. cit., *8.