A THEOREM OF FROBENIUS ON QUADRATIC FORMS*

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1. *Introduction.* A real quadratic form, in n variables,

$$\sum_{j=1}^{n} \sum_{i=1}^{n} a_{ij}x_i x_j, \qquad (a_{ij} = a_{ji}), \tag{1}$$

of rank r can always be reduced to the form

$$\sum_{i=1}^{r} c_i x_i^2, \tag{2}$$

by a real non-singular linear transformation.† The number of positive coefficients, P, and the number of negative coefficients, N, in (2) are each independent of the particular reduction used. The difference $P-N=s$ is called the signature of the form. Of the invariants P, N, $r=P+N$, s, only two are independent and any two of them form a complete system of invariants of (1) under non-singular linear transformations.

A form (1) of rank r is said to be *regularly arranged* if the x_i are so numbered that in the set

$$A_0 = 1,\ A_1 = a_{11},\ A_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}, \cdots, \tag{3}$$

$$A_r = \begin{vmatrix} a_{11} & \cdots & a_{1r} \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ a_{r1} & \cdots & a_{rr} \end{vmatrix},$$

$A_r \neq 0$, and no two consecutive A_i are zero. The following theorem is usually given for determining the invariants.

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† For the definitions and theorems quoted in this section see M. Bôcher, *Introduction to Higher Algebra*, New York, 1919, pp. 144–147; or H. Weber, *Lehrbuch der Algebra*, Braunschweig, 1895, vol. 1, pp. 255–265.

Every quadratic form may be regularly arranged, and when so arranged the signature is equal to the number of permanences minus the number of variations of sign in the sequence of A's, the A's which are zero being given signs at random.

In certain cases where the form is not regularly arranged, so that consecutive A's in the set (3) vanish, an analogous theorem holds, as Frobenius has shown.* His discussion is quite complicated. In this note we give a simple proof of the usual theorem, as well as the principal case noticed by Frobenius, based on the properties of the secular equation.

We also prove the following theorem relative to the determination of the invariants of a quadratic form from its principal minors.

If, for any quadratic form, we construct the set

$$(4)\qquad B_0 = 1,\ B_1 = \sum_i a_{ii},\ B_2 = \sum_{i,j} \begin{vmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{vmatrix}, \cdots, \quad B_n = |a_{ij}|,$$

the sums being so taken as to include each principal minor of proper order once, the index of the last non-zero term in the set is the rank, r, and the number of permanences of sign in the sequence of non-zero terms is P, so that the signature is $s=2P-r$.

2. *The Secular Equation.* We quote here a few well known theorems† concerning the equation

$$(5)\qquad L_n(x) = \begin{vmatrix} a_{11}-x & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-x & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn}-x \end{vmatrix} = 0,$$

which we shall need for our argument. Let $L_i(x)$ denote the leading i-rowed minor of $L_n(x)$, and form the sequence

* G. Frobenius, *Ueber das Trägheitsgesetz der quadratischen Formen*, Journal für Mathematik, vol. 114 (1895), pp. 187–232.

† Weber, loc. cit., vol. 1, pp. 276–279.

$$(6) \qquad L_n(x),\ -L_{n-1}(x),\ \cdots,(-1)^{n-1}L_1(x),\ (-1)^n.$$

The set (6) form a Sturm sequence for equation (5), in the sense that

If $a \leqq b$ are two numbers neither of which makes any term of the series (6) vanish, the excess of the number of variations of sign in the series (6) for $x=a$ over that for $x=b$ equals the number of real roots x between a and b, a q-fold root counting as q roots.

Each of the equations

$$(7) \qquad L_i(x) = 0, \qquad (i = 1,2,\cdots,n)$$

has all its roots real, and the roots of the $(i+1)$th equation are separated by those of the ith.

3. *Application to the Classical Criteria.* Consider any quadratic form (1) of rank r. It is always possible to reducs it to an expression of the type of (2) by an *orthogonal* traneformation on the n variables x_i.* Such a transformation will reduce the form

$$(8) \qquad \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}x_ix_j - k\sum_{i=1}^{n} x_i^2$$

to the form

$$(9) \qquad \sum_{i=1}^{r} c_ix_i^2 - k\sum_{i=1}^{n} x_i^2 .$$

The values of k which make (9), and hence (8), degenerate, are zero, counted $n-r$ times, and the r numbers c_i. Consequently these are the roots of equation (5), formed corresponding to (8). From the form of (6), we see that for $x=0$, the sequence becomes

$$(10) \qquad A_n,\ -A_{n-1},\ \cdots,\ (-1)^{n-1}A_1,\ (-1)^nA_0.$$

Now apply the theorem quoted in §2, taking for a a positive number less than any positive root of any of the equations (7), and for b, $+\infty$. The terms of the series (6) for $x=a$ cor-

* Bôcher, loc. cit., pp. 171–173.

responding to non-zero terms in the series (10) will have the same signs as these latter, while the terms corresponding to zero terms will have definite signs. As the signs of this series for $x=a$ may be obtained from those of (10) by giving the zero terms appropriate signs, and as for $x=+\infty$ all the terms of (6) have the same sign, we see that the number of positive c_i, of P, is equal to the number of variations of sign in the series (10) when the zero terms are given proper signs. Finally, since changing alternate signs of a sequence replaces permanences by variations and vice versa, we have the following theorem.

THEOREM I. *If, for any quadratic form* (1) *we set up the sequence*

$$A_0, A_1, \cdots, A_n, \tag{11}$$

and give the zero terms proper signs, the number of permanences of sign will be P.

If, in particular, $A_r \neq 0$, from the separation property of the equations (7), and the fact that $L_r(x)=0$ has now no zero root, while $L_n(x)=0$ has an $(n-r)$-fold zero root, we see that the equation $L_{r+j}(x)=0$ has precisely j zero roots. Thus as x increases through zero, the first $n-r+1$ terms of the series (6) cause $n-r$ variations to be lost. As these account for the $n-r$ zero roots, no additional variations are lost in the rest of the sequence. Also the "proper signs" for the first $n-r$ terms of (10) give rise to no variations, and hence to no permanences in (11). This gives the following theorem.

THEOREM II. *If, for any quadratic form* (1) *of rank* r *we set up the sequence*

$$A_0, A_1, \cdots, A_r, \tag{12}$$

where $A_r \neq 0$, *and give the zero terms proper signs, the number of permanences of sign will be* P.

COROLLARY 1. *If the form is regularly arranged, i. e. no two consecutive A's in* (12) *vanish, the proper signs may be chosen at random.*

For, if $A_i=0$, while $A_{i-1}\neq 0$, $A_{i+1}\neq 0$, $L_{i-1}(x)=0$ and $L_{i+1}(x)=0$ have no zero roots, and as their roots separate those of $L_i(x)=0$, this equation has a single zero root. Thus $L_i(x)$ changes sign as we pass through zero, and as no variations are lost, A_{i-1} and A_{i+1} must have opposite signs.

COROLLARY 2. *If in* (12) *at most two consecutive A's vanish, if the A's adjacent to the pair of zeros have like signs, the proper signs should lead to one permanence from this set, while if the adjacent A's have unlike signs, the proper signs should lead to two permanences from this set.*

From the separation property, the two adjacent zeros correspond to single roots. Thus the signs given to the two zeros must be such that, if both are changed, the number of permanences is not affected, which necessitates that we have, essentially,

$$+ \mp \pm + ; \qquad + \pm \pm - ;$$

or their negatives.

Considerations similar to those used to prove the corollaries may be used to restrict the possible proper signs if three or more consecutive A's vanish, but these considerations by no means determine the signs completely. In fact, if three consecutive A's vanish, the signature can not always be determined from the set (12). Thus for the form*

$$ax_2^2 + 2ax_2x_3 + 2ax_1x_4 + 2ax_3^2\,, \tag{13}$$

the sequence (12) is 1, 0, 0, 0, $-a^4$, and the signs are independent of the sign of a. However, if we renumber the variables $y_1=x_2$, $y_2=x_3$, $y_3=x_1$, $y_4=x_4$, the form becomes

$$ay_1^2 + 2ay_1y_2 + 2ay_3y_4 + 2ay_2^2\,, \tag{14}$$

* Frobenius, loc. cit., p. 199.

and the sequence (12) is 1, a, a^2, 0, $-a^4$, so that the form is now regularly arranged, and by corollary 1, $P=3$, if a is positive, while $P=1$ if a is negative.

4. *A New Criterion.* We saw in § 3 that the invariants P and N of the form (1) could be characterized as the number of positive and negative roots, respectively, of the equation (5). The criteria we developed in § 3 were based on a modified Sturm sequence for this equation. As the equation has all its roots real, we may determine the signs of the roots by Descartes' rule as the number of variations of sign in the coefficients of the equation. On expanding (5), we find

$$(15)\quad (-x)^n B_0 + (-x)^{n-1} B_1 + \cdots + (-x) B_{n-1} + B_n = 0,$$

where the B's are given by

$$(16)\quad B_0 = 1,\ B_1 = \sum_i a_{ii},\ B_2 = \sum_{i,j} \begin{vmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{vmatrix}, \cdots, \qquad B_n = |a_{ij}|,$$

the sums being so taken as to include each principal minor of proper order once. Thus for (1), P is the number of variations of sign of the coefficients of (15), or permanences of sign of the sequence (16). Since (15) has $n-r$ zero roots, $B_r \neq 0$, while all the B's following this one vanish. Collecting these results, we have the following theorem.

THEOREM III. *If, for any quadratic form* (1) *we set up the sequence*

$$(17)\quad B_0, B_1, \cdots, B_n,$$

the index of the last non-zero term in the set will be the rank, r, and the number of permanences of sign in the sequence of non-zero terms will be P, so that the signature is $s=2P-r$.

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