ON HILBERT'S THIRTEENTH PARIS PROBLEM*

BY H. W. RAUDENBUSH, JR.

At the Paris Congress in 1900, Hilbert† presented for proof the proposition that the function f of the three variables x, y, and z satisfying the equation

$$(1) \qquad f^7 + xf^3 + yf^2 + zf + 1 = 0$$

cannot be represented by the use of a finite number of continuous functions of not more than two arguments. In this note a small part of this problem is considered. We shall prove that the function f cannot have the form $F[\alpha(x, y), \beta(y, z)]$, where $F(\alpha, \beta)$, $\alpha(x, y)$ and $\beta(y, z)$ are analytic functions.

Before proceeding to the proof it is necessary to notice certain properties of the partial derivatives f_x, f_y, and f_z. They satisfy the identities

$$(2) \qquad Uf_x \equiv f^3, \qquad Uf_y \equiv f^2, \qquad Uf_z \equiv f,$$

where $U \equiv -(7f^6+3xf^2+2yf+z) \not\equiv 0$. For finite values of x, y, and z, f is finite and does not vanish. Hence U is finite and therefore the first partial derivatives cannot vanish.

In the proof we assume that

$$(3) \qquad f(x,y,z) \equiv F[\alpha(x,y),\beta(y,z)],$$

where $F(\alpha,\beta)$, $\alpha(x,y)$, and $\beta(y,z)$ are analytic functions. Since $f_x \not\equiv 0$ and $f_z \not\equiv 0$, $\alpha_x \not\equiv 0$ and $\beta_z \not\equiv 0$ for finite values of x, y, and z. The Jacobian condition for functional dependence

$$\begin{vmatrix} f_x & f_y & f_z \\ \alpha_x & \alpha_y & 0 \\ 0 & \beta_y & \beta_z \end{vmatrix} \equiv 0$$

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can then be written in the form

$$A(x,y)f_x + f_y + C(y,z)f_z \equiv 0,$$

where $A \equiv -\alpha_y/\alpha_x$ and $C \equiv -\beta_y/\beta_z$. Multiplying by U, making use of (2), and subsequently dividing by f, we get

$$A(x,y)f^2 + f + C(y,z) \equiv 0. \tag{4}$$

It is now easy to show that A is linear in x. Differentiating (4) with respect to x and separately with respect to z we find by the use of (2) that $A_x \equiv C_z$. Since C does not contain x, C_z does not contain x and hence A_x is a function of y alone.

We shall now show that A does not contain x at all. The eliminant with respect to f between (1) and (4) is an identity in x that has for its term in the highest power of x the term contributed by the expansion of A^7. This term must vanish identically and hence the coefficient of x in A must vanish identically. But if A does not contain x we have by (4) $f_x \equiv 0$. This is impossible and the assumption that f satisfies (3) leads to a contradiction.

Similarly it can be shown that f cannot have either the form $F[\alpha(x,y), \beta(x,z)]$ or the form $F[\alpha(x,z), \beta(y,z)]$; all functions being assumed analytic.

COLUMBIA UNIVERSITY