A POINT SET WHICH HAS NO TRUE QUASI-COMPONENTS, AND WHICH BECOMES CONNECTED UPON THE ADDITION OF A SINGLE POINT*

BY R. L. WILDER

If M is a set of points, and P is a point of M, then the *quasi-component* of M determined by P is the set of points common to all possible sets M_1, where $M=M_1+M_2$, M_1 contains P, and M_1 and M_2 are mutually separated sets.† If M is a connected set, then the quasi-component of M determined by P is M itself. If M is not connected, then it can be considered as the sum of its quasi-components. If a quasi-component consists of more than one point, it is called a *true quasi-component.* If all the quasi-components of M reduce to single points, that is, if M contains no true quasi-components, then M is totally disconnected (i. e., has no connected subset consisting of more than one point). However, the quasi-components of a totally disconnected set do not necessarily reduce to single points. Sierpinski has given‡ an example of a set of points N and a point P not in N, such that N has no true quasi-components, and such that the set $N+P$, although totally disconnected, contains a true quasi-component consisting of P and a certain point of N.

Knaster and Kuratowski have given§ an example of a totally disconnected set of points S, which becomes connected upon the addition of a certain point a; i. e., $S+a$

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† See F. Hausdorff, *Grundzüge der Mengenlehre*, Leipzig, 1914, p. 248. Two sets are called *mutually separated* if they have no point in common, and if neither contains a limit point of the other.

‡ *Sur les ensembles connexes et non connexes*, Fundamenta Mathematicae, vol. 2 (1921), pp. 81–95.

§ *Sur les ensembles connexes*, Fundamenta Mathematicae, vol. 2 (1921), pp. 206–255. See especially pp. 240 ff.

is a connected set. In this example, one notices that the point set S has uncountably many true quasi-components, each of which has a as a limit point.

The properties of the point sets N and S just mentioned suggest the following question: *Does there exist a point set M and a point P not in M, such that M has no true quasi-components but such that $M+P$ is connected?* It is the purpose of the present note to answer this question in the affirmative.

Consider a point set M constructed in the following manner: In the ordinary number plane, denote the points (0, 0), (2,0) and (1, 1) by A, B, and P, respectively. The interval AB of the x-axis is the sum of c (where c is the cardinal number of the continuum) mutually exclusive sets each of which is dense in AB.* Call the class of these sets X. Let $C(X)$ be a one-to-one correspondence between the elements of the class X and the real numbers $\{t\}$, $0 \leqq t \leqq 1$. Consider now the class, K, of all continua contained in triangle PAB and its interior, which have points in common with both AP† and PB, but do not contain P. Since‡ this class has the cardinal number c, there exists a one-to-one correspondence $C(K)$ between its elements and the set of real numbers $\{t\}$, $0 \leqq t \leqq 1$.

If t is any real number such that $0 \leqq t \leqq 1$, let that element of class X which corresponds to t under the correspondence $C(X)$ be denoted by X_t. Similarly, denote that element of K which corresponds to t under the correspondence $C(K)$ by K_t. Then if K_{t_1} is an element of K and X_{t_2} is an

* See Knaster and Kuratowski, loc. cit., p. 252. I wish to acknowledge here my indebtedness to this noteworthy work. It will be noticed that the suggestion for the method of attack which I employ in the present problem is contained in the construction of Knaster and Kuratowski's example (β) (pp. 245 ff.), of a connected punctiform set which is irreducible between two points.

† Hereafter in this paper, if U and V are two points, UV will denote that interval from U to V on the straight line through these two points.

‡ Cf. Knaster and Kuratowski, loc. cit., p. 253.

element of X, these two sets will be said to correspond to one another if and only if $t_1 = t_2$.

If T is any point of the interval AB, whose abscissa is x, denote the interval PT by l_x. For every real number t, $0 \leqq t \leqq 1$, there corresponds a set of points M_t defined as follows: For every interval l_x, where x is the abscissa of a point of X_t, let m_x be the point of minimum ordinate of the set of points common to K_t (the set which corresponds to X_t) and l_x; the set of all points $\{m_x\}$ constitutes the set M_t. For every real number t, $0 \leqq t \leqq 1$, there corresponds a definite set M_t, and no two of these sets have a point in common.

Denote by M' the set of all points p, such that p is contained in some set M_t. Denote by M the set $M' - M_0$.

1. The set of points M is totally disconnected, and, moreover, has no true quasi-components. For let a and b be two points of M. Let l_{x_1} and l_{x_2} be those intervals of the set $\{l_x\}$ that contain a and b, respectively. Since no l_x contains more than one point of M, x_1 and x_2 are unequal, say $x_1 < x_2$. As X_0 is dense in AB, there exists a point of X_0 whose abscissa, x_3, satisfies the relation $x_1 < x_3 < x_2$. The interval l_{x_3} contains no point of M, by definition. Let the set of all points of M which lie on intervals l_x for which $x < x_3$ be denoted by M_1, and the set of all points of M which lie on lines l_x for which $x > x_3$ be denoted by M_2. Clearly M_1 and M_2 are mutually separated sets containing a and b, respectively. Hence M is totally disconnected and has no true quasi-components.

2. The set of points $M+P$ is connected. For suppose it is not connected. Then

$$M+P = M_1 + M_2,$$

where M_1 contains P and M_1 and M_2 are mutually separated non-vacuous sets. Let Q be a point of M_2. By a theorem due to Knaster and Kuratowski,* there exists a continuum

* Loc. cit., Theorem 37.

H^* which separates the plane between P and Q, and contains no point of $M+P$.

Let that interval of the set $\{l_x\}$ which contains Q be denoted by PF, where F is the intersection of this interval with AB. Then the arc $PAFQ=PA+AF+FQ$, together with PQ, forms a simple closed curve J. Denote the interior of J by R. By Lemma 2 of my paper *A connected and regular point set which has no subcontinuum*,† there exists a subcontinuum H' of H which is a subset of $J+R$ and has at least one point in common with each of the arcs $PAFQ$ and PQ. Since neither P nor Q is a point of H', it is clear that H' cannot lie wholly on PF. Let g be the smallest number such that l_g contains a point of H'. Clearly $0 \leqq g < f$, where f is the abscissa of F. Denote the point of AB whose abscissa is g by G. Let D and E be points of H' on PG and PF, respectively. Then $AD+H'+EB$ forms a continuum of class K, say K_{t_1}. If $t_1=0$, let B' be an interior point of the interval PB, and let K_{t_1} be the continuum $AD+H'+EB'$. Then $t_1 \neq 0$, and the set M_{t_1} is a subset of $M+P$. As $g<f$, and X_{t_1} is dense on AB, there exists a number e such that $g<e<f$ and that point of AB whose abscissa is e belongs to X_{t_1}. Then that point of M_{t_1} determined by the intersection of l_e and K_{t_1} is necessarily a point of H'. That is, H' and M have points in common. This is impossible, since H' is a subset of H and the latter set has no point in common with M. Hence the supposition that $M+P$ is not connected leads to a contradiction.

Thus by 1. and 2. the set of points $M+P$ is an example furnishing an affirmative answer to the question raised at the beginning of this note.

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* A continuum is a closed and connected set which contains more than one point. A continuum H is said to separate the plane between two points P and Q if P and Q do not lie in the same domain complementary to H, and neither lies in H.

† See Transactions of this Society, vol. 29(1927), pp. 332–340.

I shall close with one further observation as to the properties of the set M. Mazurkiewicz calls* a set M quasi-connected if for every point m of M there corresponds a positive number λ such that there does not exist any division of M into two mutually separated sets M_1 and M_2 such that M_1 contains m and the diameter of M_1 is less than λ. He gives† an example of a quasi-connected set which contains no true quasi-components. That the set M constructed above is another example of such a set is easily shown; indeed, the value of λ may be taken *uniformly* equal to unity for all points of M.

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A SIMPLE METHOD FOR NORMALIZING TCHEBYCHEFF POLYNOMIALS AND EVALUATING THE ELEMENTS OF THE ALLIED CONTINUED FRACTIONS‡

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1. *Introduction.* Consider a system

$$P_n(x), \qquad (n = 0,1,2, \cdots), \tag{1}$$

of orthogonal, but not normal, Tchebycheff polynomials corresponding to a given (finite or infinite) interval (a, b) with the characteristic function $p(x)$. The corresponding *normalized* system of polynomials will be denoted by

$$\phi_n(p\,;\,x) \equiv \phi_n(x) = a_n(p)\left[x^n - S_n(p)x^{n-1} + \cdots\right], \quad (n = 0,1, \cdots, a_n > 0). \tag{2}$$

We have, then,

$$\int_a^b p(x)\phi_m(x)\phi_n(x)dx = \begin{cases} 0, m \neq n, \\ 1, m = n. \end{cases} \tag{3}$$

* *Sur les ensembles quasi-connexes*, Fundamenta Mathematicae, vol. 2 (1921), pp. 201–205.

† Loc. cit.

‡ Presented to the Society, April 16, 1927.