CORRADO SEGRE

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There is a pronounced rise and fall in the tide of mathematical interest in different countries and at different times. This is well shown in the case of geometry. In Great Britain and Ireland, in the Victorian era, Hamilton and Clifford, Salmon and Casey, Cayley and Sylvester, constituted a veritable English-Irish geometric school. With the deaths of these men, and the decline in the fashion for algebraic invariants, all of this efflorescence seemed to wither away, till the University of Oxford appointed to the Savillian professorship of geometry a man who had attained high distinction in the analytical theory of numbers. In France, the name of Chasles exercised, at one time, a fascination which we perhaps do not quite understand today. Darboux, with his masterly *Théorie Générale des Surfaces*, raised to himself a monument more enduring than bronze, but no geometric school arose therefrom, and the center of gravity of mathematical interest in France today is certainly not in the field of geometry. In Germany, the fluctuation has been distinctly less. In Italy, the interest in geometry has produced notable results. The classical differential geometry owes an immense debt to Beltrami and Bianchi; the Riemannian geometers take as their starting point the absolute calculus of Ricci and Levi-Civita; as for the modern birational geometry, that is almost a monopoly of Italian mathematicians and a few others like Macaulay and Snyder, who have been strongly influenced by Italian thought.

What is the reason for this state of affairs? It would be equally wrong to ascribe it entirely to chance or to seek a recondite explanation in the psychology of peoples. The real ground is to be found in the presence or absence of great teachers of geometry. Salmon was a great teacher indeed, but Dublin was not a good center for distributing the pure milk of the word. Darboux was a marvellous lecturer, but he lacked the personal qualities necessary to inspire youth. It is of Jules Tannery, not of Darboux, that the French mathematicians of today say: "C'était notre père à nous tous." In Germany, on the other hand, Klein's extraordinary inspiration was felt quite as much in geometry as elsewhere; Reye created an enthusiasm for pure projective geometry that went almost to excess; the present geometric activity in Hamburg and Bonn is directly traceable to the influence of Study. The geometric "risorgimento" in Italy is the result of the efforts of certain great teachers such as Cremona, Battaglini, D'Ovidio, and the subject of the present sketch.

Corrado Segre was born in Saluzzo, August 20, 1863*, and received his laureate at Turin when less than twenty years of age. He never left

* In preparing the present account, I have consulted, in addition to Segre's own writings, Castelnuovo, Rendiconti dei Lincei, (5a), vol. 33[2]

the Piedmontese capital thereafter, except for short periods; his active life was passed there as assistant and as professor. At some period during the early part of his career, he had a severe attack of brain fever brought on by overwork. He complained sadly that his capacity for labor was never the same thereafter. When one reflects that the number of his published titles is 128, it fatigues the imagination to speculate as to what he might have accomplished had he retained his early strength. He died unexpectedly, after an operation, on May 18, 1924.

The connecting thread which runs through Segre's work is the projective geometry of n dimensions. In the hands of a man of small grasp, this might easily degenerate into barren formalism. Not so in his case. He never lost sight of the relations between a general formulation and concrete problems, and never failed to familiarize himself with other methods of treating the question in hand, besides those which he personally preferred. Let us trace the line of his thought. The simplest figures in any space are the linear sub-spaces. Suppose first that we have a finite number of such. The obvious problems are those of incidence. The beginner has little idea of the complications which quickly arise in such questions. Is it at all obvious that in four-space exactly five planes will intersect six given lines in general position? Segre treats them in two papers separated by a considerable interval of time.* Next to linear spaces we have quadratic varieties. Segre's thesis is devoted to a study of the hyperquadric in n-space.† A single hyperquadric is of slight mathematical interest compared with the study of two such figures and the classification of their various types of intersection. The beautiful plane curve which Darboux has called a cyclic and the English and Irish have described as a bicircular quartic, is the stereographic projection on the projective plane of the intersection of the Gauss sphere with a general quadric surface. Ascending one dimension, the general cyclide or quartic surface with a double conic may be reached in a similar way by a projection in four-space. Segre devoted one of his earliest and largest papers to the classification of such surfaces.‡ Going up two more dimensions, the line-geometry of Plücker and Klein consists in identifying the lines of a projective three-space, with the points of a hyperquadric in five-space. The intersections of two hyperquadrics will correspond to the lines of a quadratic complex. Segre studies the general quadratic complex and some particular varieties, with special reference to the singular elements, most fruitfully by this method.§

(1924); Baker, Journal of the London Society, vol. 1 (1926), and especially Loria, Annali di Matematica, (4), vol. II (1925). Here will be found a complete bibliography of Segre's work.

* Rendiconti di Palermo, vol. 2 (1888); Rendiconti dei Lincei, (5), vol. 9^2 (1900).

† Torino Memorie, (2), vol. 36 (1883).

‡ Mathematische Annalen, vol. 34 (1884).

§ There are a number of papers on this topic. See Loria, loc. cit., pp. 14–15.

The problem of classifying the intersections of two hyperquadrics is best solved by the use of the elementary divisors of Weierstrass. An almost identical algebraic analysis will lead to the classification of linear transformations and their fixed elements. This problem also received Segre's attention.*

Reverting to linear spaces, Segre had the curious but fruitful idea of studying those systems of points which are in one to one correspondence with groups of points, one in each of a number of given linear spaces. The simplest example of this is the perfect correspondence between the individual points of a general quadric and pairs of points of two given lines. The case which particularly interested Segre was that where there were two conjugate imaginary linear varieties.† These considerations led him to another field to which, for a few years, he gave considerable attention. If a complex space of any number of dimensions be given, there are two questions which immediately suggest themselves. How can the points of this space, and of various loci therein, be represented by real geometrical figures? What are the geometrical characteristics of the loci of points whose coordinates are functions of a certain number of *real* parameters? For instance, how shall we best represent all the complex points of a plane? Given a system of points in a plane whose coordinates depend analytically on two real parameters, how shall we determine whether they generate a curve? Segre laid the foundation for the future study of the second of these problems and added considerably to our knowledge of the first.‡ He thereby established contact with Study, whom he subsequently met at the Heidelberg Congress in 1904, and with whom he retained cordial relations for many years. It must be confessed, however, that few mathematicians have been found who cared to enter this which Segre calls "Un nuovo campo di ricerche geometriche." Even so competent a critic as Baker§ fails completely to grasp the real significance of this part of Segre's work. Perhaps he felt some regret that so little was done to extend geometrical science in these directions. The present writer feels that regret keenly.

Let us return to linear spaces and suppose that we have not a finite but an infinite number of them. We are led, as the case may be, to problems of algebraic or of differential geometry. Suppose that we have a one-parameter system of lines in our space. If it be an algebraic system, it will correspond to a curve in 5-space, and so have a definite genus. But these lines will generate an algebraic surface with various genera, arithmetic, geometric, etc. On the other hand if the system be not algebraic, but still analytic, the lines will generate a surface. Is this surface de-

* Rendiconti dei Lincei, (3), vol. 19 (1884); Torino Memorie, (2), vol. 37 (1885); Journal für Mathematik, vol. 100 (1886).

† Rendiconti di Palermo, vol. 5 (1891).

‡ Torino Atti, vol. 25 (1889), vol. 26 (1890); Mathematische Annalen, vol. 40 (1891).

§ Loc. cit., p. 270.

velopable, and, if not, what can be said of the other system of asymptotic lines besides that formed by the rectilinear generators? Segre studied questions of each of these types, the algebraic ones in his earlier years, the differential ones later on.* In fact the subject of projective differential geometry was that to which he gave most attention during the later years of his life. He made the acquaintance of Wilczynski at Heidelberg in 1904 and thoroughly appreciated the importance of his work. Yet an outsider cannot help feeling some regret that there still appears a considerable gulf between the Italian doctrine of projective differential geometry, developed by Segre, Bompiani, Fubini and others, and the American contributions of Wilczynski and Gabriel Green, of whom, by the way, no Italian seems to have heard the name.

It is time to turn from Segre's scientific writings to five important articles of an expository character which he produced at various times. The longest, and by far the most important of these was his article in the Mathematical Encyclopedia on the geometry of n-space.† The thought that he must one day complete this, depressed his spirits at times for a good many years, for he was one who took his responsibilities seriously, and he felt in honor bound to put the thing through. Complete it he finally did, thus earning the gratitude of geometers for many years to come. To quote Baker‡

"For completeness of detail, breadth of view, and generous recognition of the work of a host of other writers, this must remain for many years a monument of the comprehensiveness of the man."

In 1891, while still a very young teacher, he wrote for the benefit of his pupils "Su alcuni indirizzi nelle investigazioni geometriche"§ which was reprinted many years later in this Bulletin|| under the title *On some tendencies in geometrical investigations*. It is a cheering message on the whole, almost as helpful to the young mathematicians of our time and country, as to those of the epoch and place for which it was originally intended. The point which Segre enforces most strongly is that at all costs the beginner must learn to choose really significant subjects. To acquire the instinct to do this, he should study the mathematical classics. He must especially avoid trivial questions and perfectly obvious extensions or generalizations. The young geometer should also be familiar with the leading methods and results of modern analysis, and welcome help from every quarter, regardless of whether his own preference is for synthetic, algebraic, arithmetical or transcendental methods. With regard to rigor, Segre recommends the geometer to be as rigorous as he can, and to own up like a man when he makes use of methods of doubtful parentage.

* See especially Rendiconti di Palermo, vol. 30 (1910).

† Encyklopädie der mathematischen Wissenschaften, III C 7, pp. 769–972.

‡ Loc. cit., p. 271.

§ Rivista di Matematica, vol. 1 (1891).

|| Vol. 10 (1903–04).

At the same time he does not consider that geometry is necessarily a succession of δ, ϵ processes, and is not shocked at the judicious use of the convenient phrase "in general".

Very close in spirit to this is his address delivered many years later at the Heidelberg Congress *La geometria d'oggidi e i suoi legami coll' analisi.** His thesis is that geometry and analysis deal with what is essentially the same subject matter, but with different emphasis, terminology and esthetic aim, a view which probably is even more acceptable today than it was when first expressed. He lays special stress on various branches of algebraic geometry, particularly those properties of figures which are unaltered by birational transformation. Near the close, in speaking of the classic memoir of Brill and Noether† he remarks "Tutta una scuola di geometri italiani riconosce nella Memoria di Brill e Noether il suo punto di partenza." This statement falls short of the truth today, through excess of modesty. The present tendency is to take as point of departure two remarkable, and mutually complementary articles which appeared a dozen years before Segre's address, and which deal with the birationally invariant properties of curves.‡ The first by Bertini follows classical algebraic methods, which are set forth with astonishing clearness and elegance. The second by Segre, which shows a remarkable range of mathematical reading from Kronecker to Castelnuovo, follows whenever possible the fundamental methods of the projective geometry of n-space. What has n-space to do with algebraic plane curves? A very great deal. Suppose that we have on a plane curve, not one point, nor a group of points, but a system of such groups, each group containing n points, while the system depends linearly on r independent parameters. These numbers will be birationally invariant. Let us suppose that there are no fixed points common to all of the groups, and that the system is not an involution, i. e., every group containing a general point does not automatically contain certain others. Lastly we may assume that the system is cut by $r+1$ linearly independent adjoint curves $f_0, f_1, \cdots, f_r$. If we write

$$\rho X_i = f_i(x_1, x_2, x_3),$$

we shall have a transformation which carries our plane curve into a space curve of order n in a space of r dimensions, and the groups of our system into the intersections of our new curve with the hyperplanes of its space. For instance, if we have a plane quartic curve with two double points, the conics through these points will meet the curve in a three-parameter system of groups of four points, and will enable us to transform the curve birationally into the elliptic quartic of three-space. This is another aspect of the stereographic projection of a space quartic into a bicircular plane quartic already discussed.

* Published in volume 2 of the Transactions of that Congress; and also in Rendiconti di Palermo, vol. 19 (1905).

† *Über die algebraischen Funktionen und ihre Anwendung in der Geometrie,* Mathematische Annalen, vol. 7 (1873).

‡ Annali di Matematica, (2), vol. 22 (1893).

We have just used the words "adjoint curves." An adjoint to a given algebraic plane curve is a second which has the multiplicity $s_i - 1$ or more, wherever the original curve has the multiplicity s_i. When the singular points of the original curve are "ordinary", that is to say, when the tangents are all distinct, this definition is perfectly satisfactory. But when the tangents fall together in groups, the singular point may become a very ugly customer, and call for extremely careful treatment in determining the deficiency of the curve and the conditions imposed on adjoints. The problem of determining the number of intersections of two curves absorbed by a common singularity has attracted the attention of geometers from H. J. S. Smith to Enriques. It is the subject of Segre's last long expository paper,* the method of treatment being a careful study of the structure of the resolvent.

Behind Segre the geometer and the expositor, there remained always Segre the teacher and friend. An interesting figure he was in the lecture room. Of medium height and frail, half seated on the end of the table, gesticulating rapidly with his left hand, while his right whirled his watch-chain about with astonishing angular velocity, his whole being was thrown into the task of driving home the essentials of what he had in mind. The subject matter of his course was new every year. There was no limit to the amount of care and patience which he would bestow on one of his pupils. To one who apologized for taking so much of his valuable time with trivial difficulties, he wrote: "Surtout, il ne faut jamais hésiter à me consulter quand vous trouverez une difficulté. Il n'y a que les ignorants et les paresseux qui n'ont jamais de difficultés."

He had a sympathetic, sensitive nature, a spiritual quality, which impressed all who had the privilege of his friendship. Most truly Castelnuovo wrote of him†:

"Corrado Segre ebbe una virtù preziosa concessa a pochissimi eletti: la virtù di rendere migliori tutti coloro che lo avvicinavano. Chi ha potuto vivere qualche tempo nella sua intimità ne ha sentito influenza benefica per tutta la vita."

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* Giornale di Matematiche, vol. 36 (1897).

† Loc. cit., p. 359.