INVARIANTS OF A PORISTIC SYSTEM
OF TRIANGLES*

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1. *Introduction.* In the last quarter of a century consider-able work has been done on various systems of triangles.† Several fixed circles have been discovered which are as-sociated with poristic systems of triangles. However, so far as the writer knows, no attempt has been made to determine all the invariant curves of any degree for any system of triangles. The present paper determines the invariant curves of the first and second degrees for a poristic system of triangles which has a fixed incircle and a fixed circumcircle.

2. *General Considerations.* Let there be a triangle ABC with a fixed incircle, center I, and a fixed circumcircle, center O. If we consider OI as the X-axis and I as the origin, the equations of the sides of the triangle may be written in the normal form as follows:

$$(1) \quad \begin{cases} BC: & x\cos A_1 + y\sin A_1 - r = 0, \\ CA: & x\cos A_2 + y\sin A_2 - r = 0, \\ AB: & x\cos A_3 + y\sin A_3 - r = 0, \end{cases}$$

where r denotes the radius of the incircle.

If $A'B'C'$ be any other triangle of the system the equations of its sides are

$$(2) \quad \begin{cases} B'C': & x\cos A_1' + y\sin A_1' - r = 0, \\ C'A': & x\cos A_2' + y\sin A_2' - r = 0, \\ A'B': & x\cos A_3' + y\sin A_3' - r = 0. \end{cases}$$

Let $P(x', y')$ be any point in the plane of the triangles. Then the distances of P from the sides (1) and (2) are

$$(3) \quad \alpha_i = x'\cos A_i + y'\sin A_i - r, \qquad (i = 1, 2, 3),$$

$$(4) \quad \alpha_i' = x'\cos A_i' + y'\sin A_i' - r, \qquad (i = 1, 2, 3),$$

respectively.

Eliminating x' and y' from (3) and (4), we obtain

$$(5) \qquad \alpha_i' = \frac{|\alpha_1, \sin A_2, 1|}{|\cos A_1, \sin A_2, 1|} \cos A_i'$$

$$+ \frac{|\cos A_1, \alpha_2, 1|}{|\cos A_1, \sin A_2, 1|} \sin A_i' - r.$$

The angle A_2' determines A_1' and A_3'. Hence α_i' depends upon α_i and A_2', if A_1, A_2 and A_3 remain fixed.

Equations (5) then determine a one-parameter group of transformations, and by the Lie Theory of one-parameter groups we have for an invariant

$$(6) \qquad \sum_{i=1}^{3} \left[\frac{\partial \alpha_i'}{\partial A_2'} \right]_{A_2'=A_2} \frac{\partial}{\partial \alpha_i} \equiv 0.$$

Equation (6) for the particular case represented by equation (5) becomes

$$(6') \qquad \sum_{i=1}^{3} \left[- \frac{|\alpha_1, \sin A_2, 1|}{|\cos A_1, \sin A_2, 1|} \sin A_i \frac{dA_i}{dA_2} \right.$$

$$\left. + \frac{|\cos A_1, \alpha_2, 1|}{|\cos A_1, \sin A_2, 1|} \cos A_i \frac{dA_i}{dA_2} \right] \frac{\partial}{\partial \alpha_i} \equiv 0.$$

If we choose for the triangle ABC the special position where B lies on the line OI, and to the right of I, equation (6') will be much simplified without loss of generality. In fact it becomes

$$(7) \quad \left\{ \begin{array}{l} \left[(2\alpha_2 - \alpha_3 - \alpha_1) \cos^2 \frac{B}{2} \cdot \frac{dA_1}{dA_2} \right. \\[2ex] \quad + (\alpha_1 - \alpha_3)\left(\sin \frac{B}{2} + 1 \right)\left(\sin \frac{B}{2} \right)\frac{dA_1}{dA_2} \bigg] \frac{\partial}{\partial \alpha_1} \\[2ex] \quad + (\alpha_3 - \alpha_1)\left(\sin \frac{B}{2} + 1 \right) \frac{\partial}{\partial \alpha_2} \\[2ex] \quad + \left[-(2\alpha_2 - \alpha_3 - \alpha_1) \cos^2 \frac{B}{2} \cdot \frac{dA_3}{dA_2} \right. \\[2ex] \quad + (\alpha_1 - \alpha_3)\left(\sin \frac{B}{2} + 1 \right) \sin \frac{B}{2} \cdot \frac{dA_3}{dA_2} \bigg] \frac{\partial}{\partial \alpha_3} \equiv 0, \end{array} \right.$$

where $dA_3/dA_2 = dA_1/dA_2 = R/(R-d)$, in which R denotes the radius of the circumcircle, and $d = OI$.

3. *A First Degree Invariant.* Let

$$\text{(8)} \qquad\qquad l\alpha_1 + m\alpha_2 + n\alpha_3 = 0$$

represent any line in the plane of the triangle ABC. If we apply the differential operator (7) to the line (8), we obtain

$$\text{(9)} \quad [(k' - k)l - k''m + (k' + k)n]\alpha_1 + (2kl - 2kn)\alpha_2$$
$$+ [-(k' + k)l + k''m - (k' - k)n]\alpha_3 \equiv 0,$$

where

$$k = \cos^2 \frac{B}{2} \cdot \frac{dA_1}{dA_2};$$

$$k' = \left(\sin \frac{B}{2} + 1\right) \sin \frac{B}{2} \cdot \frac{dA_1}{dA_2};$$

$$k'' = \sin \frac{B}{2} + 1.$$

In order that (9) shall be satisfied we must have

$$\text{(10)} \quad \left\{ \begin{array}{r} (k' - k)l - k''m + (k' + k)n = 0, \\ l - n = 0, \\ -(k' + k)l + k''m - (k' - k)n = 0. \end{array} \right.$$

The matrix of the set of equations (10) is of rank 2, hence there is one solution for l, m, n, other than zeros. A solution may then be written $l = l$, $m = (2k'/k'')l$, $n = l$. But $2k'/k'' = 1$ and the equation of the single invariant line of the system is

$$\alpha_1 + \alpha_2 + \alpha_3 = 0.$$

This is the equation of the line in which the bisectors of the exterior angles of the triangle meet the opposite sides; hence we may state the following theorem.

THEOREM I. *In a poristic system of triangles, which has a fixed incircle and a fixed circumcircle, the line determined by the points of intersection of the bisectors of the exterior angles of the triangle with the opposite sides is a fixed line.*

4. *Second Degree Invariants.* The equation of any curve of the second degree in the plane of the triangle ABC may be written

(11) $l\alpha_1^2 + m\alpha_2^2 + n\alpha_3^2 + 2p\alpha_1\alpha_2 + 2s\alpha_1\alpha_3 + 2t\alpha_2\alpha_3 = 0.$

If we apply the differential operator (7) to the equation (11) we obtain

$$[l\alpha_1 + p\alpha_2 + s\alpha_3][(2\alpha_2 - \alpha_3 - \alpha_1)k + (\alpha_1 - \alpha_3)k']$$

$$+ (m\alpha_2 + p\alpha_1 + t\alpha_3)(\alpha_3 - \alpha_1)k''$$

(12)

$$+ [-(2\alpha_2 - \alpha_3 - \alpha_1)k + (\alpha_1 - \alpha_3)k'][n\alpha_3 + s\alpha_1 + t\alpha_2]$$

$$\equiv 0.$$

In order that equation (12) shall be valid we must have the following set of equations.

$$l(k' - k) - pk'' + s(k' + k) = 0,$$

$$p = t,$$

$$n(k' - k) - tk'' + s(k' + k) = 0,$$

(13)

$$mk'' - 2nk - p(k' + k) + 2sk - t(k' - k) = 0,$$

$$l(k' + k) - n(k' + k) - pk'' + tk'' = 0,$$

$$2lk - mk'' + p(k' - k) - 2ks + (k' + k)t = 0.$$

The matrix of equations (13) is of rank 4. Hence there is a single infinity of solutions for the set of equations. This set of solutions may be written in terms of l and p, as follows:

$$n = l, \ t = p, \quad s = \frac{R - d}{R}p + \frac{d}{R}l, \ m = \frac{R + d}{R}l - \frac{d}{R}p.$$

Equation (12) may therefore be written in the form

$$l\alpha_1^2 + \left(\frac{R+d}{R}l - \frac{d}{R}p\right)\alpha_2^2 + l\alpha_3^2 + 2p\alpha_1\alpha_2$$

(14)

$$+ 2\left(\frac{R-d}{R}p + \frac{d}{R}l\right)\alpha_1\alpha_3 + 2p\alpha_2\alpha_3 = 0.$$

By finding the polar of the incenter with respect to the family (14) we obtain a result which may be stated in the following theorem.

THEOREM II. *The first polar of the incenter with respect to the family* (14) *is the line* $\alpha_1 + \alpha_2 + \alpha_3 = 0$.

Also if we write down the condition that the system (14) shall represent two lines we find that it is necessary that $l = p$.*

A substitution of this condition in (14) gives $(\alpha_1 + \alpha_2 + \alpha_3) = 0$. Hence the straight lines of the system (14) consist of the first degree invariant counted twice.

If we set down the condition that equation (14) represent a circle we find

$$p = l\left(\frac{R+d}{R}\right).$$

A substitution of this condition in (14) gives for the equation of the circle the expression

(15)
$$\alpha_1^2 + \frac{2r}{R}\alpha_2^2 + \alpha_3^2 + 2\left(\frac{R+d}{R}\right)\alpha_1\alpha_2$$
$$+ 2\left(\frac{2r+d}{R}\right)\alpha_1\alpha_3 + 2\left(\frac{R+d}{R}\right)\alpha_2\alpha_3 = 0.$$

The center of this circle lies on OI. And since we have chosen as our triangle of reference the triangle whose vertex B lies on the line OI, the equation of OI is

(16) $\alpha_1 - \alpha_3 = 0$.

Also from the triangle itself we have

(17) $\alpha_1 + \left(\dfrac{R-d}{R}\right)\alpha_2 + \alpha_3 = \left(\dfrac{R-d}{R}\right)(R + d + r)$.

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* See Whitworth, *Modern Analytical Geometry*, p. 244.

If we solve equations (15), (16), (17) for α_2 we find

$$\alpha_2 = \frac{d(R + d + r) \pm [Rd(R + d)\,(R + d + r)]^{1/2}}{d}.$$

The half sum of the two values of α_2 gives the distance of the center of the circle (15) from the side AC of the triangle. This distance is $R+d+r$. Hence we have the following theorem.

THEOREM III. *The center of the invariant circle* (15) *lies on the circumcircle of the variable system of triangles.*

The half difference of the values of α_2 gives the radius ρ of the circle (15). Consequently

$$\rho = \frac{[Rd(R + d)\,(R + d + r)]^{1/2}}{d}.$$

Using this result we may easily prove the following theorem.

THEOREM IV. *The line* $\alpha_1+\alpha_2+\alpha_3=0$ *is the radical axis of the incircle and the circle* (15).

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