$t = t_1 s_1$. Since $(t_1 s_1)^{n_1} = s_1^{n_1}$ is in (s), and n_1 is prime to the order of s_1, s_1 is a power of s. Thus t_1, as well as t, corresponds to t' in the isomorphism of G with $G/(s)$; but t_1 and t' are of the same order n_1. Every element of $G/(s)$ whose order is a divisor of m/m_1 corresponds to an element of G whose order is a divisor of m. It follows that t', and hence every element of $G/(s)$ whose order divides n, is commutative with every element whose order divides m/m_1. Hence $G/(s)$, being of order $< mn$, contains an invariant subgroup of order n. The corresponding subgroup of G, being of order $m_1 n < mn$, also contains an invariant subgroup of order n.

Thus in all cases G contains a subgroup of order n. Similarly G contains a subgroup of order m. G is evidently the direct product of these two subgroups.

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A CUBIC CURVE CONNECTED WITH TWO TRIANGLES

BY H. BATEMAN

1. *Introduction.* If ABC, XYZ are two triangles, a cubic curve Γ_3 may be associated with them as follows.* Let (PQ, RS) denote the point of intersection of the lines PQ, RS; then Γ_3 is the locus of a point O such that (OA, YZ), (OB, ZX), (OC, XY) are collinear and also the locus of a point O for which (OX, BC), (OY, CA), (OZ, AB) are collinear. In fact when one set of three points is collinear the other set of three is also collinear. Take ABC as triangle of reference and let the points X, Y, Z have coordinates (x_1, x_2, x_3), (y_1, y_2, y_3), (z_1, z_2, z_3) respectively, then if (α, β, γ) are current coordinates

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* H. Grassmann, *Die lineale Ausdehnungslehre*, 1844, p. 226. The corresponding quartic surface connected with two tetrahedra is mentioned by H. Fritz, Pr. Ludw. Gymn. Darmstadt [reference taken from Jahrbuch der Fortschritte der Mathematik, vol. 21 (1889), p. 725] and by C. M. Jessop, *Quartic Surfaces*, Cambridge, 1916, p. 189.

the condition for the collinearity of the second sets of three points may be expressed in the form

$$(\gamma x_1 - \alpha x_3)(\alpha y_2 - \beta y_1)(\beta z_3 - \gamma z_2)$$
$$= (\alpha x_2 - \beta x_1)(\beta y_3 - \gamma y_2)(\gamma z_1 - \alpha z_3).$$

This equation represents a cubic curve passing through the points* A, B, C, X, Y, Z, (BC, YZ), (CA, ZX), (AB, XY), (CY, BZ), (AZ, CX), (BX, AY). There is evidently only one cubic curve passing through all these points, and since a cubic through all these points is obtained for the locus when the three points (OA, YZ), (OB, ZX), (OC, XY) are collinear, Γ_3 must be the locus in both cases.

This is easily verified analytically. If X_1, X_2, etc. denote the co-factors of the constituents x_1, x_2, etc. in the determinant

$$\Delta = \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix},$$

the equation of the locus in the first case may be expressed in the form

$$\begin{vmatrix} \beta X_2 + \gamma X_3 & -\beta X_1 & -\gamma X_1 \\ -\alpha Y_2 & \alpha Y_1 + \gamma Y_3 & -\gamma Y_2 \\ -\alpha Z_3 & -\beta Z_3 & \alpha Z_1 + \beta Z_2 \end{vmatrix} = 0,$$

and the equivalence of the two equations is easily verified with the aid of a number of identities, one of which is

$$2X_1 Y_2 Z_3 - Y_1 X_3 Z_2 - Z_1 X_2 Y_3$$
$$= \Delta(2x_1 y_2 z_3 - y_1 x_3 z_2 - z_1 x_2 y_3)$$
$$= \Delta(x_1 X_1 + y_2 Y_2 + z_3 Z_3 - \Delta).$$

2. *Salmon's Projective Invariant.* By a well known theorem due to Salmon the cross ratio of the four tangents to a cubic from the point on the curve is constant and represents

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* The properties of this configuration of 12 points and some degenerate cases of the cubic curve are discussed by H. M. Taylor, Proceedings of the London Society, (1), vol. 28 (1897), pp. 545–555.

an invariant of the curve. In the present case this invariant
will be an invariant of the two triangles* and it seems worth
while to calculate it. A direct calculation is laborious when
use is made of the equation of the curve in one of the forms
already given; the work may be simplified, however, by notic-
ing that the four lines BC, YZ, BZ, CY form a complete quadri-
lateral whose six vertices lie on the curve. Denoting the equa-
tions of these lines by $\alpha = 0$, $\theta = 0$, $\phi = 0$, $\psi = 0$ respectively,
the equation of the curve may be written in the form

$$\frac{A}{\alpha} + \frac{B}{\theta} + \frac{C}{\phi} + \frac{D}{\psi} = 0,$$

where the identical relation between α, θ, ϕ, ψ is

$$\alpha + \theta + \phi + \psi = 0,$$

and

$$A = y_2 z_1^2 X_2^2 Z_3, \qquad B = y_1^2 z_3 X_3^2 Y_2,$$
$$C = y_1^2 z_1^2 X_1 \Delta, \qquad D = x_1 X_2^2 X_3^2 .$$

If λ is one of the cross ratios of the four tangents, λ is connect-
ed with the coefficients A, B, C, D by the relation†

$$(\lambda + 1)^2 (\lambda - 2)^2 (\lambda - \tfrac{1}{2})^2 I^3 = 27(\lambda^2 - \lambda + 1)^3 J^2,$$

where I and J, the usual invariants of a biquadratic equation,
are given by the equations

$$12I = \Theta^2 - 48ABCD, \qquad 216J = \Theta[72ABCD - \Theta^2],$$
$$I^3 - 27J^2 = A^2 B^2 C^2 D^2 [\Theta^2 - 64ABCD],$$

where

$$\Theta^2 = A^2 + B^2 + C^2 + D^2 - 2BC - 2CA - 2AB$$
$$- 2AD - 2BD - 2CD.$$

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* A complete set of invariants of two triangles is given in a paper by
D. D. Leib, Transactions of this Society, vol. 10 (1909), p. 361. In our case
there is a correspondence between the vertices of the triangles and the in-
variant theory is a little different.

† The coefficient m in the canonical equation $x^3 + y^3 + z^3 + 6mxyz = 0$ is
connected with I and J by the relation
$$\frac{64m^3(m^3 - 1)^3}{I^3} = \frac{(8m^6 + 20m^3 - 1)^2}{27J^2} .$$
See H. Hilton, *Plane Algebraic Curves*, Oxford, 1920, p. 235.

Substituting the expressions for A, B, C, D, we find that

$$\Theta = y_1^2 z_1^2 X_2^2 X_3^2 \left[(x_3 y_1 z_2 - x_2 y_3 z_1)^2 - 4\Delta x_1 y_2 z_3 - 4 X_1 Y_2 Z_3 \right],$$

$$ABCD = y_1^4 z_1^4 X_2^4 X_3^4 \left[x_1 X_1 y_2 Y_2 z_3 Z_3 \Delta \right].$$

The nature of the cubic is thus seen to depend on the value of the single invariant

$$K = \frac{\left[(x_3 y_1 z_2 - x_2 y_3 z_1)^2 - 4\Delta x_1 y_2 z_3 - 4 X_1 Y_2 Z_3 \right]^2}{x_1 y_2 z_3 X_1 Y_2 Z_3 \Delta}.$$

The cubic has a double point when this invariant is either infinite or 64. When $\Theta = 0$ we have $J = 0$ and the four tangents form a harmonic pencil. If A, B, C, Y, and Z are given, the locus of X when $\Theta = 0$ is a conic. This, however, is not the complete locus when the tangents form a harmonic pencil because J vanishes also when $\Theta^2 = 72ABCD$ or $K = 72$. It should be noticed that the expression K remains the same when we replace each constituent in the determinant by its co-factor. We have in fact an identity of type

$$X_3 Y_1 Z_2 - X_2 Y_3 Z_1 = \Delta(z_3 Z_3 - y_2 Y_2 + x_3 X_3 - x_2 X_2)$$
$$= \Delta(x_3 y_1 z_2 - x_2 y_3 z_1),$$

with the aid of which the preceding remark is easily verified.

Quantities that possess this property may be called *invariants* of the determinant. To find such quantities we may commence by finding pseudo-invariants, that is, quantities which retain the same form except for a power of the determinant. The quantities $x_1 X_1$, $x_2 X_2$, $x_3 X_3$, $y_1 Y_1$, $y_2 Y_2$, $y_3 Y_3$, $z_1 Z_1$, $z_2 Z_2$, $z_3 Z_3$ all possess this last property when considered as functions of the constituents x_1, x_2, etc. and a similar remark may be made in the case of a determinant of higher order. For the general case of a determinant of order n I do not know definitely how many pseudo-invariants of degree n in the constituents are linearly independent. A consideration of the cases $n = 2, 3$, and 4 leads to the conclusion that this number N may be given by the formula

$$N = (n - 1)^2 + 1.$$

When $n = 2$, it is evident that $N = 2$. When $n = 3$ it appears that $N = 5$, for each of the nine quantities x_1X_1, x_2X_2, x_3X_3, y_1Y_1, y_2Y_2, y_3Y_3, z_1Z_1, z_2Z_2, z_3Z_3 can be expressed as a linear function of the five linearly independent quantities x_1X_1, x_2X_2, y_1Y_1, y_2Y_2, Δ.

In the case of a determinant of the fourth order

$$\begin{vmatrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \\ t_1 & t_2 & t_3 & t_4 \end{vmatrix},$$

the problem is more complex because in addition to the pseudo-invariants x_sX_s etc. there are pseudo-invariants of type $(y_2z_3 - y_3z_2)(x_1t_4 - x_4t_1)$.

Adopting the notation

$$\begin{pmatrix} yz \\ 14 \end{pmatrix} = (y_2z_3 - y_3z_2)(x_1t_4 - x_4t_1) = \begin{pmatrix} x\,t \\ 23 \end{pmatrix},$$

we observe that the relations of type

$$y_1Y_1 = x_2X_2 + \begin{pmatrix} x\,t \\ 12 \end{pmatrix} + \begin{pmatrix} z\,x \\ 12 \end{pmatrix} - \begin{pmatrix} xy \\ 31 \end{pmatrix} - \begin{pmatrix} xy \\ 14 \end{pmatrix},$$

$$y_2Y_2 = x_1X_1 + \begin{pmatrix} z\,x \\ 12 \end{pmatrix} + \begin{pmatrix} x\,t \\ 12 \end{pmatrix} + \begin{pmatrix} xy \\ 24 \end{pmatrix} + \begin{pmatrix} xy \\ 23 \end{pmatrix},$$

$$y_3Y_3 = x_4X_4 + \begin{pmatrix} x\,t \\ 34 \end{pmatrix} + \begin{pmatrix} z\,x \\ 34 \end{pmatrix} - \begin{pmatrix} xy \\ 31 \end{pmatrix} - \begin{pmatrix} xy \\ 23 \end{pmatrix},$$

$$y_4Y_4 = x_3X_3 + \begin{pmatrix} x\,t \\ 34 \end{pmatrix} + \begin{pmatrix} z\,x \\ 34 \end{pmatrix} - \begin{pmatrix} xy \\ 14 \end{pmatrix} - \begin{pmatrix} xy \\ 24 \end{pmatrix}$$

indicate that all the pseudo-invariants of type y_sY_s, z_sZ_s, t_sT_s can be expressed in terms of the pseudo-invariants x_1X_1, x_2X_2, x_3X_3, x_4X_4 and the 18 quantities of type

$$\begin{pmatrix} yz \\ 14 \end{pmatrix}.$$

It would be wrong, however, to suppose that there are 22 linearly independent pseudo-invariants of degree four, for there are the six identities

$$x_1X_1 + x_4X_4 = \begin{pmatrix} xy \\ 23 \end{pmatrix} + \begin{pmatrix} zx \\ 23 \end{pmatrix} + \begin{pmatrix} xt \\ 23 \end{pmatrix},$$

$$x_2X_2 + x_4X_4 = \begin{pmatrix} xy \\ 31 \end{pmatrix} + \begin{pmatrix} zx \\ 31 \end{pmatrix} + \begin{pmatrix} xt \\ 31 \end{pmatrix},$$

$$x_3X_3 + x_4X_4 = \begin{pmatrix} xy \\ 12 \end{pmatrix} + \begin{pmatrix} zx \\ 12 \end{pmatrix} + \begin{pmatrix} xt \\ 12 \end{pmatrix},$$

$$x_2X_2 + x_3X_3 = \begin{pmatrix} xy \\ 14 \end{pmatrix} + \begin{pmatrix} zx \\ 14 \end{pmatrix} + \begin{pmatrix} xt \\ 14 \end{pmatrix},$$

$$x_3X_3 + x_1X_1 = \begin{pmatrix} xy \\ 24 \end{pmatrix} + \begin{pmatrix} zx \\ 24 \end{pmatrix} + \begin{pmatrix} xt \\ 24 \end{pmatrix},$$

$$x_1X_1 + x_2X_2 = \begin{pmatrix} xy \\ 34 \end{pmatrix} + \begin{pmatrix} zx \\ 34 \end{pmatrix} + \begin{pmatrix} xt \\ 34 \end{pmatrix},$$

three relations of type

$$\Delta = \begin{pmatrix} xy \\ 34 \end{pmatrix} + \begin{pmatrix} xy \\ 14 \end{pmatrix} + \begin{pmatrix} xy \\ 24 \end{pmatrix} + \begin{pmatrix} xy \\ 23 \end{pmatrix} + \begin{pmatrix} xy \\ 31 \end{pmatrix} + \begin{pmatrix} xy \\ 12 \end{pmatrix},$$

three relations of type

$$\Delta = \begin{pmatrix} yz \\ 12 \end{pmatrix} + \begin{pmatrix} zx \\ 12 \end{pmatrix} + \begin{pmatrix} xy \\ 12 \end{pmatrix} + \begin{pmatrix} xt \\ 12 \end{pmatrix} + \begin{pmatrix} yt \\ 12 \end{pmatrix} + \begin{pmatrix} zt \\ 12 \end{pmatrix},$$

and the relation $x_1X_1 + x_2X_2 + x_3X_3 + x_4X_4 = \Delta$. The number of linearly independent pseudo-invariants thus seems to be ten.

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