# ON CONTINUITY IN SEVERAL VARIABLES*

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The following theorem on the continuity of a function of several variables is contained implicitly in a theorem on the existence of solutions of differential equations by Carathéodory.† It is of a general nature and independent of the context in which it is found. It is, therefore, worth while isolating and signalizing it.

THEOREM. *Hypotheses*: (1) $f(x, [y])$ *is defined in the domain* $R: a<x<b,\ y_0-k<[y]<y_0+k$, *where* $[y]\equiv(y_1, \cdots, y_n)$.

(2) *For each* $[y]$ *in* $R$, $f_x'(x, [y])$ *exists almost everywhere on* $a<x<b$, *and is summable on* $a<x<b$.

(3) $|f_x'(x, [y])|\leqq M(x)$, *where* $M(x)$ *is summable on* $a<x<b$.

(4) *For every* $x$ *on* $a<x<b$, $f(x, [y])$ *is continuous in* $[y]$ *at* $y_1=y_0$.

*Conclusion*: $f(x, [y])$ *is continuous in* $(x, [y])$ *at* $(x, [y_0])$ *where* $x$ *is on* $a<x<b$.

PROOF. Let us put $F_1\equiv f(x+h,[y_0+k_i])$, where $|k_i|\leqq k$; $F_2\equiv f(x,[y_0+k_i])$; $F_3\equiv f(x,[y_0])$. From hypothesis (2), we have

$$F_1 - F_2 = \int_x^{x+h} f_t'(t, [y_0 + k_i])dt,$$

which becomes by hypothesis (3),

$$(1)\qquad |F_1 - F_2| \leqq \int_x^{x+h} M(t)dt.$$

Also, from hypothesis (4), it follows that given a positive $\epsilon$, there exists a positive number $k_{xy_0}'$, such that

$$(2)\qquad |F_2 - F_3| < \epsilon, \quad \text{for} \quad |k_i| < k_{xy_0}'.$$

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† *Vorlesungen ueber Reelle Funktionen*, Leipzig, 1918, p. 678, Satz 5.

On the other hand, since $F_1 - F_3 \equiv F_1 - F_2 + F_2 - F_3$. it follows that

$$|F_1 - F_3| \leqq |F_1 - F_2| + |F_2 - F_3|.$$

By applying (1) and (2) to this inequality, we obtain the result $f(x, [y])$ is continuous in $(x, [y])$ at $(x, [y_0])$, where $x$ is any value on $a<x<b$.

Let $F(x, y)$ be defined in the interior of the square $S$: $a<x, y<b$, and have the following properties, (1) for a fixed $y$, it is summable in $x$ on $a<x<b$, (2) $|F(x, y)| \leqq K(x)$, for all values of $y$ on $a<y<b$, where $K(x)$ is summable on $a<x<b$, (3) for every fixed $x$ except a null set, $F(x, y)$ is continuous in $y$. Consider

$$f(x,y) = \int_a^x F(t,y)dt.$$

From a well known theorem concerning an indefinite integral we have that $f(x, y)$ is continuous in $x$ for $y$ fixed. From a theorem first stated by Carathéodory,* $f(x, y)$ is continuous in $y$ for fixed $x$. *From the theorem stated above*, $f(x, y)$ is continuous in $(x, y)$ for all points in $R$.

The conclusion of the preceding paragraph is of importance in showing that if a system of first order differential equations which satisfy the conditions of Carathéodory's† existence theorem be solved by the method of successive approximations, not only will the solutions be continuous functions of the independent variable and the parameter when both vary arbitrarily (as given by Carathéodory's theorem) but also the approximating functions will have continuity of this same kind.‡

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* Loc. cit., p. 639.

† Loc. cit., p. 678.

‡ For a pair of linear first order differential equations, see Sturdivant, *Properties of second order linear systems with Lebesgue integrable coefficients*, offered to the Transactions of this Society.