Now (14) is identical with (3) defining the space transversality of J. Hence our theorem.

The theorem is easily extended to an m space immersed in an n space. Since a Riemann space of m dimensions can always be immersed in a flat space of at most $\frac{1}{2}m(m+1)$ dimensions, we have an immediate proof of the Gauss theorem for a Riemann space of any number of dimensions. For it is obvious that the transversality of the length integral in a flat n-space is the orthogonality of lineal elements to $(n-1)$-elements with the same base point, and evidently the section of this transversality by any m-spread contained in the n-flat is the orthogonality of lineal to $(m-1)$-elements in the m-spread.

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ON THE EXTENSION OF A METHOD OF BRIOT AND BOUQUET FOR THE REDUCTION OF SINGULAR POINTS*

BY B. O. KOOPMAN

In a classical memoir,† Briot and Bouquet gave a method by means of which the differential equation

$$\frac{dx}{X(x,y)} = \frac{dy}{Y(x,y)}$$

could be reduced to a simple standard form in the neighborhood of an analytic singular point, i. e., a point at which $X(x, y)$ and $Y(x, y)$ are analytic, but vanish simultaneously. Although the method fails to be directly applicable to certain special cases, it has shown itself to be of sufficient

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† JOURNAL DE L'ECOLE POLYTECHNIQUE, vol. 21, p. 161. See also Picard, *Traité d'Analyse*, Paris, Gauthier-Villars, 1908, vol. 3, p. 34.

value* to warrant its extension to systems of higher order, an extension which forms the subject of the present note.

For clarity of exposition, we shall deal with the system of the second order

$$\frac{dx}{X(x,y,z)} = \frac{dy}{Y(x,y,z)} = \frac{dz}{Z(x,y,z)} \tag{1}$$

for which the origin is an analytic singular point:

$$\left\{\begin{aligned} X &= \sum_{\alpha',\beta',\gamma'}^{1-\infty} A'_{\alpha'\beta'\gamma'} x^{\alpha'} y^{\beta'} z^{\gamma'}, \\ Y &= \sum_{\alpha'',\beta'',\gamma''}^{1-\infty} A''_{\alpha''\beta''\gamma''} x^{\alpha''} y^{\beta''} z^{\gamma''}, \\ Z &= \sum_{\alpha''',\beta''',\gamma'''}^{1-\infty} A'''_{\alpha'''\beta'''\gamma'''} x^{\alpha'''} y^{\beta'''} z^{\gamma'''}, \\ &\alpha^{(i)} + \beta^{(i)} + \gamma^{(i)} \geqq 1. \end{aligned}\right. \tag{2}$$

We wish to find the integral curves which approach the origin, and have there the form

$$x = ut^p, \qquad y = vt^q, \qquad z = wt^r. \tag{3}$$

Here u, v, w are functions of the parameter t, distinct from zero when $t=0$, and p, q, r, positive integers. When these expressions are substituted in equation (1), we get

$$\begin{aligned} &\frac{updt + tdu}{\sum_{\alpha',\beta',\gamma'} A'_{\alpha'\beta'\gamma'} u^{\alpha'} v^{\beta'} w^{\gamma'} t^{(\alpha'-1)p+\beta' q+\gamma' r+1}} \\ &= \frac{vqdt + tdv}{\sum_{\alpha'',\beta'',\gamma''} A''_{\alpha''\beta''\gamma''} u^{\alpha''} v^{\beta''} w^{\gamma''} t^{\alpha'' p+(\beta''-1) q+\gamma'' r+1}} \\ &= \frac{wrdt + tdw}{\sum_{\alpha''',\beta''',\gamma'''} A'''_{\alpha'''\beta'''\gamma'''} u^{\alpha'''} v^{\beta'''} w^{\gamma'''} t^{\alpha''' p+\beta''' q+(\gamma'''-1) r+1}}. \end{aligned} \tag{4}$$

* See J. Malmquist, *Sur les points singuliers des équations différentielles*, ARKIV FÖR MATEMATIK, ASTRONOMI OCH FYSIK, vol. 15 (1921).

Suppose that p, q, r have been so chosen that the terms of lowest degree in t in these denominators are of the same degree, δ. Then we must have, for a certain set, $\alpha_0', \beta_0', \cdots, \gamma_0'''$, of the exponents,

$$
(5)\qquad \begin{cases} (\alpha_0' - 1)p + \beta_0' q + \gamma_0' r = \delta - 1, \\ \alpha_0'' p + (\beta_0'' - 1)q + \gamma_0'' r = \delta - 1, \\ \alpha_0''' p + \beta_0''' q + (\gamma_0''' - 1)r = \delta - 1. \end{cases}
$$

Our first problem is, then, to determine values of α', β', $\cdots$, γ''' in equations (2) such that there exist positive integers, p, q, r which have the following properties: (i) p, q, r satisfy (5). (ii) If

$$
A'_{\alpha_1'\beta_1'\gamma_1'}, \quad A''_{\alpha_1''\beta_1''\gamma_1''} \quad A'''_{\alpha_1'''\beta_1'''\gamma_1'''}
$$

are any other non-vanishing coefficients of terms of equation (2), and if $\alpha_0', \beta_0', \cdots, \gamma_0'''$ are replaced by α_1', β_1', $\cdots$, γ_1''', respectively, in (5), the left-hand members take on values which are not less than $\delta - 1$.

To find these quantities, we proceed geometrically. In the 3-dimensional space of rectangular coordinates (ξ, η, ζ), plot the following three classes of points:

(I) The points $(\alpha' - 1, \beta', \gamma')$, corresponding to all the non-vanishing $A'_{\alpha'\beta'\gamma'}$'s present in (2).

(II) The points $(\alpha'', \beta'' - 1, \gamma'')$, corresponding to the $A''_{\alpha''\beta''\gamma''}$'s.

(III) The points $(\alpha''', \beta''', \gamma''' - 1)$, corresponding to the $A'''_{\alpha'''\beta'''\gamma'''}$'s.

All these points lie within or upon the solid angle formed by the three quarter-planes

$$
\begin{aligned} \xi &= -1, & \eta &\geqq -1, & \zeta &\geqq -1; \\ \xi &\geqq -1, & \eta &= -1, & \zeta &\geqq -1; \\ \xi &\geqq -1, & \eta &\geqq -1, & \zeta &= -1. \end{aligned}
$$

Unless X, Y, Z are polynomials, the points will spread out to infinity. Let Π be a plane which cuts all three edges of the solid angle, and which determines with the faces a tetrahedron not containing any of the points (I), (II), or (III) in its interior. Suppose that there lies on Π at least one point $(\alpha_0' - 1, \beta_0', \gamma_0')$, one point $(\alpha_0'', \beta_0'' - 1, \gamma_0'')$, and one point $(\alpha_0''', \beta_0''', \gamma_0''' - 1)$. Then it is clearly possible to take as direction components of the normal to Π three relatively prime positive integers, which we will call p, q, r. The equation of Π will then be

$$(6) \qquad p\xi + q\eta + r\zeta = \rho .$$

I say that the $\alpha_0', \beta_0', \cdots, \gamma_0''', p, q, r$ determined in the above manner have the requisite properties (i), (ii). That they satisfy the first requirement is seen from the fact that $(\alpha_0' - 1, \beta_0', \gamma_0')$, etc., lie on Π, (taking $\delta = \rho + 1$). As for the second requirement, let $\alpha_1', \beta_1', \cdots, \gamma_1'''$ be any set of values represented in (2). By construction, we know that the points $(\alpha_1' - 1, \beta_1', \gamma_1')$, etc., do not lie on the same side of Π as the vertex $(-1, -1, -1)$. It follows, on applying the rudiments of analytic geometry to the equation (6) and the points in question, that

$$(\alpha_1' - 1)p + \beta_1' q + \gamma_1' r \geqq p = \delta - 1, \text{ etc.}$$

i. e., requirement (ii) is satisfied.

We are now ready to complete the reduction. Let u_0, v_0, w_0 be the values which u, v, w take on when t equals zero. Then, when the common factor t^δ is removed from equations (4) and t set equal to zero, they yield

$$(7) \qquad \frac{u_0 p}{SA'_{\alpha'\beta'\gamma'} u_0^{\alpha'} v_0^{\beta'} w_0^{\gamma'}} = \frac{v_0 q}{SA''_{\alpha''\beta''\gamma''} u_0^{\alpha''} v_0^{\beta''} w_0^{\gamma''}} = \frac{w_0 r}{SA'''_{\alpha'''\beta'''\gamma'''} u_0^{\alpha'''} v_0^{\beta'''} w_0^{\gamma'''}} .$$

Here S denotes a summation extended over those values of $\alpha', \beta', \cdots, \gamma'''$ which satisfy equations (5). It will in general be possible to select arbitrarily one of the quantities u_0, v_0, w_0, when values of the others will be determined by equations (7).

Everything has been done so far on the assumption that a plane Π exists. We now add some further hypotheses, which are fulfilled in general:

(a) There exists a set of quantities u_0, v_0, w_0, none of which are zero, which satisfy equations (7).

(b) The solutions u_0, v_0, w_0 are simple roots of (7).

(c) None of the quantities $SA'_{\alpha'\beta'\gamma'}\ u_0^{\alpha'}v_0^{\beta'}w_0^{\gamma'}$, etc., are zero for the solutions u_0, v_0, w_0 in question.

Suppose that it is desired to regard x as the independent variable. Take $u=1$ in equations (3) (this amounts to replacing t by the new variable $u^{1/\nu}t$), and perform the same substitutions as before, obtaining equations (4) and (7), only with $u=u_0=1$, $du=0$. Now set $v=v_0+\sigma_1$, $w=w_0+\sigma_2$; our equations will reduce to

$$(8)\qquad \begin{cases} t\dfrac{d\sigma_1}{dt} = a_1t + b_1\sigma_1 + c_1\sigma_2 + \phi_1(t,\sigma_1,\sigma_2), \\ t\dfrac{d\sigma_2}{dt} = a_2t + b_2\sigma_1 + c_2\sigma_2 + \phi_2(t,\sigma_1,\sigma_2), \end{cases}$$

where the expansions of ϕ_1 and ϕ_2 start with terms of the second degree or higher in t, σ_1, σ_2.

We have thus performed the reduction, corresponding to that of Briot and Bouquet, for the general system of the second order. Its extension to the nth order is carried out in a precisely similar manner, with the aid of the simplest notions of hypergeometry. In case certain of our hypotheses fail to be realized, we can still obtain results from the method of reduction; it is useless to elaborate this point further here, as the methods can be worked out in analogy with those for the first order.*

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* See Picard, loc. cit.; Malmquist, loc. cit.