NOTE ON A PROBLEM IN APPROXIMATION WITH AUXILIARY CONDITIONS*

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Let $\rho(x)$ and $f(x)$ be two given functions of period 2π, the former bounded and measurable, with a positive lower bound, the latter, for simplicity, continuous. Among all trigonometric sums $T_n(x)$, of given order n, there is one and just one for which the value of the integral

$$\int_0^{2\pi} \rho(x)\,[f(x) - T_n(x)]^2 dx \tag{1}$$

is a minimum. If the weight function $\rho(x)$ is identically 1; it is a matter of familiar knowledge that the minimum is reached when $T_n(x)$ is the partial sum of the Fourier series for $f(x)$. A considerable amount of attention has been given recently to the problem of the convergence of the minimizing sum $T_n(x)$ toward $f(x)$, as n becomes infinite, under the generalized conditions that result from the admission of an arbitrary weight function.†

Let $x_1, \cdots, x_N$ be N values of x in the interval $0 \leqq x < 2\pi$. The problems of the preceding paragraph may be further varied by admitting to consideration only such sums $T_n(x)$ as satisfy the conditions

$$T_n(x_i) = f(x_i), \qquad (i = 1, 2, \cdots, N), \tag{2}$$

and inquiring after the minimum of the integral (1) subject to these auxiliary conditions. It is understood that the given value of n is large enough so that the conditions (2) can be

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† Cf. e.g., D. Jackson, *Note on the convergence of weighted trigonometric series*, this BULLETIN, vol. 29 (1923), pp. 259–263, where further bibliographical references will be found; also D. Jackson, *A generalized problem in weighted approximation*, TRANSACTIONS OF THIS SOCIETY, vol. 26 (1924), pp. 133–154.

fulfilled; for this it is sufficient that $n \geqq \frac{1}{2}(N-1)$. There is no essential novelty in the proof of the existence and uniqueness of the sum which yields the minimum. The notation $T_n(x)$ being restricted henceforth to this "approximating sum," it is the purpose of the following lines to discuss the convergence of $T_n(x)$ toward $f(x)$, as n increases without limit. The question is not trivial, even if $\rho(x) \equiv 1$. It is quite distinct from the conventional problems of interpolation, inasmuch as N is fixed, and does not increase with n.

For each value of $n(\geqq \frac{1}{2}(N-1))$, let a continuous function $\varphi_n(x)$ be defined, having ϵ_n as an upper bound for its absolute value, and such that

$$\varphi_n(x_i) = 0\ , \qquad (i = 1, 2, \cdots, N). \tag{3}$$

Let $\tau_n(x)$ be the approximating sum of the nth order for $\varphi_n(x)$; that is, the sum which minimizes the integral

$$\int_0^{2\pi} \rho(x)\,[\varphi_n(x) - \tau_n(x)]^2 dx\ ,$$

subject to the conditions

$$\tau_n(x_i) = 0\ , \qquad (i = 1, 2, \cdots, N). \tag{4}$$

Exactly as in the absence of the restrictions (3), (4), it may be shown* that

$$|\varphi_n(x) - \tau_n(x)| \leqq k\epsilon_n\sqrt{n}\ ,$$

where k is independent of n (expressible, in fact, in terms of the ratio of the upper and lower bounds of $\rho(x)$, and independent of anything else). The new conditions call for notice only to the extent of the observation that a trigonometric sum which vanishes identically comes within the requirements of (4). Furthermore, it is recognized at once that if $\varphi_n(x)$ is defined by the relation

$$\varphi_n(x) = f(x) - t_n(x)\ ,$$

where $t_n(x)$ is a trigonometric sum of the nth order *taking on the same values as* $f(x)$ *at the points* $x_1, x_2, \cdots, x_N$, then $\tau_n(x)$ and $T_n(x)$ are related by the identity†

* D. Jackson, this BULLETIN, loc. cit.

† Cf. this BULLETIN, loc. cit., p. 261.

$$\tau_n(x) = T_n(x) - t_n(x) \ ,$$

so that $f(x) - T_n(x) = \varphi_n(x) - \tau_n(x)$ indentically, and

$$|f(x) - T_n(x)| \leqq k\epsilon_n\sqrt{n} \ .$$

The formulation of sufficient conditions for convergence is reduced then to the determination of the order of magnitude of ϵ_n, the measure of the accuracy with which $f(x)$ can be uniformly approached by trigonometric sums $t_n(x)$ such that

$$(5) \qquad t_n(x_i) = f(x_i) \ , \qquad (i = 1, 2, \cdots, N).$$

Let $y_1, y_2 \cdots, y_N$ be any N numbers subject to the conditions $|y_i| \leqq 1$, $i = 1, 2, \cdots, N$. If N is even, let x_0 be a point in $(0, 2\pi)$ distinct from $x_1, x_2, \cdots, x_N$, and let $y_0 = 1$. Let $t(x)$ be the trigonometric sum of order $\frac{1}{2}(N-1)$ or $\frac{1}{2}N$, according as N is odd or even, which takes on the values $[y_0]$, $y_1, \cdots, y_N$ at the points $[x_0]$, $x_1, \cdots, x_N$. Let g be the maximum of $|t(x)|$. This g is a continuous function* of $y_1, y_2, \cdots, y_N$, and has a maximum G, as the y's range over all admissible values. If 1 is replaced by η as upper bound for the absolute values of the y's, the greatest possible absolute value of the corresponding $t(x)$ is $G\eta$.

Now suppose it is known that for each $n \geqq \frac{1}{2}N$ there is a trigonometric sum $\bar{t}_n(x)$, of the nth order, satisfying everywhere the relation

$$|f(x) - \bar{t}_n(x)| \leqq \eta_n \ ,$$

but not further specially restricted at the points $x_1, \cdots, x_N$. Let

$$y_i = f(x_i) - \bar{t}_n(x_i) \ , \qquad (i = 1, 2, \cdots, N),$$

and let $t(x)$ be determined as above, for this set of y's. Then $|t(x)| \leqq G\eta_n$, where G is independent of n (being dependent only on $x_1, \cdots, x_N$). The determination of

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* Explicitly, as is well known,

$$t(x) = \sum_{i=1}^{N} y_i \frac{\sin\frac{1}{2}(x-x_1) \cdots \sin\frac{1}{2}(x-x_{i-1})\sin\frac{1}{2}(x-x_{i+1}) \cdots \sin\frac{1}{2}(x-x_N)}{\sin\frac{1}{2}(x_i-x_1) \cdots \sin\frac{1}{2}(x_i-x_{i-1})\sin\frac{1}{2}(x_i-x_{i+1}) \cdots \sin\frac{1}{2}(x_i-x_N)}$$

when N is odd, the initial index 1 being replaced by 0 when N is even.

$t(x)$ is different for different values of n, but each $t(x)$ is itself a trigonometric sum of order $\frac{1}{2}N$ at most. The identity

$$t_n(x) = \bar{t}_n(x) + t(x)$$

defines a sum of the nth order such that the conditions (5) are fulfilled, and such that

$$|f(x) - t_n(x)| \leqq (1+G)\eta_n .$$

As the factor $1+G$ is independent of n, this means that the order of the attainable approximation is not affected by the imposition of the restrictions (5).

In particular, if $\omega(\delta)$ is the maximum of $|f(x')-f(x'')|$ for $|x'-x''| \leqq \delta$, and if $\lim_{\delta \to 0} \omega(\delta)/\sqrt{\delta} = 0$, sums $\bar{t}_n(x)$ will exist* such that $\lim_{n\to\infty} \eta_n \sqrt{n} = 0$, and there will consequently be sums $t_n(x)$ such that $\lim_{n\to\infty} \epsilon_n \sqrt{n} = 0$. We may state the result as a theorem, identical in form with the one found when the auxiliary conditions (2) are omitted.

THEOREM. *The sum $T_n(x)$ will converge uniformly to the value $f(x)$ for $n \to \infty$, provided that*

$$\lim_{\delta \to 0} \omega(\delta)/\sqrt{\delta} = 0 .$$

It is readily seen that essentially the same treatment can be carried through if $[f(x) - T_n(x)]^2$ in (1) is replaced by $|f(x) - T_n(x)|^m$, for any value of $m>1$; the condition for convergence is that $\lim_{\delta\to 0} \omega(\delta)/\delta^{1/m} = 0$. The discussion can be further extended in various ways that need not be elaborated here.

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* Cf. this BULLETIN, loc. cit., p. 261.