THE FEBRUARY MEETING IN NEW YORK

The two hundred forty-seventh regular meeting of the Society was held at Columbia University, New York City, on February 27. The attendance included the following forty-six members of the Society:

C. R. Ballantine, J. P. Ballantine, A. A. Bennett, Brinkmann, B. H. Camp, Alonzo Church, L. W. Cohen, Fiske, Fite, Fort, R. M. Foster, Frink, Gafafer, Gill, Gronwall, C. C. Grove, Hedlund, Hille, Himwich, B. F. Kimball, Kormes, Langford, Langman, Lefschetz, Littauer, MacColl, Meder, Norwood, Pedersen, Pfeiffer, Pogo, C. L. Poor, Rainich, R. G. D. Richardson, Ritt, Schelkunoff, Seely, Siceloff, M. H. Stone, J. H. Taylor, Tyler, Veblen, Weida, Weiss, Whittemore, W. A. Wilson.

There was no meeting of the Council or of the Trustees.

Professor Bennett presided at the morning session, relieved by Professor Lefschetz. Dr. Gronwall presided at the afternoon session.

Titles and abstracts of the papers read at this meeting follow below. The papers of Adams, Cleveland, Dodd, Eisenhart, Franklin, Gehman, Hille (second paper), Hollcroft, Lubben, Mathewson, Moore, Walsh, and Whyburn were read by title. Mr. Smith was introduced by Professor Lefschetz, and Mr. Cleveland and Mr. Whyburn by Professor R. L. Moore.

1. Professor C. L. Poor: *A direct determination of the relativity formula for the "line element."*

One of the fundamental equations of relativity dynamics is that for the "line element." This is a particular solution of Einstein's general equations and has been derived by Schwarzschild, Eddington, and others by algebraic processes involving the formulas of Riemann and Christoffel. This formula gives, for a body moving in a gravitational field, the value of a minute interval of local, relativity, time (ds) in terms of the equivalent interval of classical time (dt), and in terms of the coordinates and velocities of the body. In his original papers in the ANNALEN DER PHYSIK, Einstein gives definite formulas for the relations between measurements of time and space as made in classical units and in the special units of the general theory of relativity. Now the "line element" can be derived directly from these fundamental formulas, or assumptions, of Einstein by simple methods of analytical geometry and the ordinary formulas of calculus. This method of deriving the equation brings out clearly its essential feature as a formula for transforming time intervals from one set of units to another set.

2. Professor C. L. Poor: *Note on the relativity formula for orbital motion.*

The differential equation for the orbital motion of a planet in relativity dynamics differs from the Newtonian equation by a small term which, it is claimed, indicates the necessity for a minute correction to Newton's law of gravitation. This relativity equation, however, is derived directly from the formula for the "line element" and from two equations for the geodesic. And the orbital formula must of necessity involve and depend upon the assumptions, or premises, contained in these three equations, and it can involve no conclusion contrary to these premises. Now, one of the assumptions thus used is the ordinary formula of classical mechanics for the constancy of areas, but expressed in relativity local time. The second equation of the geodesic combined with that of the "line element" involves and depends upon the law of inverse squares. Thus the orbital equation of relativity dynamics gives the motion of a planet under the Newtonian law of inverse squares in terms of "local" relativity units of measurement in place of the standard units of classical mechanics. The difference between the Einsteinian and the Newtonian formulas of orbital motion is one of units of measurement only, not one of different laws of gravitation.

3. Mr. G. Y. Rainich: *Radiation in curved space-time.*

The vector q previously introduced by the author and interpreted as indicating the presence of radiation is now considered to be zero throughout space-time with the exception of singular lines, where it is finite and along which it is constant. The possibility of treating such lines with the aid of residues is indicated. These singular lines represent light impulses and differ essentially from those representing particles of matter. The projection of q on a time direction gives the frequency of light for observers at rest in the space perpendicular to this time direction. The relation of this vector q to the "vector quantum" is discussed. The points of view of wave motion and of emission seem to be reconciled. The singular lines considered are also characterized by the vanishing of ω, also introduced before, which means that the electric and magnetic vectors are perpendicular and of equal length. A general form of the Riemann tensor which allows a two-parameter group of rotations is introduced. It is shown that in spaces which have such a Riemann tensor the vector q vanishes in regular domains and Maxwell's equations are satisfied.

4. Professor J. H. Taylor: *Concerning a curvature tensor.* Preliminary report.

In this paper a tensor is developed which is associated with the system of equations $d^2x^i/ds^2+C^i(x, dx/dt)=0$, $i=1, 2, \cdots, n$, where the functions

C^i are required to be homogeneous of degree two in the derivatives. The tensor obtained is a formal generalization of the curvature tensor of the geometry of paths, as it reduces to the latter when the functions C^i are quadratic forms in dx/ds.

5. Professor S. Lefschetz: *On the singularities of algebraic surfaces and varieties.*

This paper deals with a new proof, much simpler than any yet given, of the theorem on the representation of the neighborhood of any singular point of an algebraic surface by means of a finite number of sets of power series. The theorem is also extended to any algebraic variety.

6. Mr. P. A. Smith: *The approximation of curves and surfaces by algebraic curves and surfaces.*

It is shown in this paper that a simple closed curve in a plane may be represented by a converging polynomial series. The polynomials may be so chosen that by equating successive sums to zero, a sequence of non-singular algebraic ovals converging uniformly to the curve is obtained. This result is extended to a restricted class of closed $(n-1)$-dimensional manifolds in S_n.

7. Dr. T. H. Gronwall: *On the remainder term in the binomial series.* Second paper.

In the author's first paper with the above title, a form of the remainder terms was set up, and simple properties established for it inside the unit circle. This investigation is now extended to the entire plane cut along the real axis from $+1$ to $+\infty$.

8. Dr. J. P. Ballantine: *A numerical method of solving differential equations with a remainder.*

It is shown how to form a table of difference quotients where the first order entries are alternately derivatives and differences. After the table is started each step consists in guessing the value of the next difference quotient of order three. From this, by the laws of the table, the next value of y is obtained. Now, knowing the value of x and y, the value of y' is obtained from the differential equation. Again, by the laws of the table, the next third order difference quotient is obtained. By comparing the guessed entry and the resulting obtained entry, and by observing a simple rule, the guess may be corrected, the correction bringing about minor corrections in the dependent entries, so as to make the column of third order entries appear regular. An expression is obtained for an absolute upper bound of the error in a solution obtained in this way, involving the fourth order entries in the table and $F_y(x, y)$. In practice this method compares favorably with others in the following respects: (1) ease of computation; (2) possible length of stride; (3) cumulative error; (4) availability of the resulting function for interpolation; (5) flexibility for various required degrees of accuracy. The only disadvantage is with respect to starting.

9. Dr. J. P. Ballantine: *A lemma in n-space.*

In p and ϵ are two positive numbers, then there exists an integer n such that in all spaces of n or more dimensions if two vectors are drawn from the origin to random points of the unit hypersphere, then the probability is less than p that the absolute value of the cosine of the angle between the two vectors is greater than ϵ.

10. Dr. J. P. Ballantine: *Measures of precision with weaker assumptions than usual.*

If n measurements of a quantity are made, the n errors constitute the components of the error vector. It is assumed that (1) the norm of the error vector divided by n approaches a limit as n increases; and (2) if the error vector is produced to cut the unit hypersphere, it cuts in a random point. From these two assumptions and the lemma of the preceding abstract, the usual formulas of precision are derived. The usual assumption in place of (2), expressed in terms of the error vector, in effect limits the point in which it cuts the hypersphere to one of $n!$ points, one corresponding to each of the $n!$ permutations of n errors of a "normal distribution."

11. Professor Einar Hille: *On Laguerre's series. Three notes.*

The formal expansions of the Fourier type in terms of generalized Laguerre polynomials

$$L_n^\alpha(x) = \frac{1}{n!}\, x^{-\alpha} e^x \frac{d^n}{dx^n}\left[x^{n+\alpha} e^{-x}\right]$$

of various classes of functions are investigated as to convergence and summability. In the first note the existence of null expansions is pointed out, which are either convergent or summable (C, k) with $k > \alpha + \frac{1}{2}$. Further a theorem on Abel summability, due to Wigert in the case $\alpha = 0$, is extended to the general case $\alpha > -1$. In the second note the convergence problem is attacked under rather general assumptions on the function which is to be expanded. The special case $\alpha = 0$ is treated in the third note with still less restrictive assumptions.

12. Professor Einar Hille: *A class of reciprocal functions. Supplementary note on the convergence of Hermitian series.*

Results communicated to the Society by the author at the summer meeting of 1925 are applied to the convergence problem of Hermitian series. The conditions are slightly more general than those previously found by Weyl and Cramér.

13. Professor L. P. Eisenhart: *Geometries of paths for which the equations of the paths admit $n(n+1)/2$ independent linear first integrals.*

Spaces of constant Riemannian curvature of n dimensions are characterized by the property that the equations of the geodesics admit linear first integrals involving $n(n+1)/2$ arbitrary constants. The present paper, as evidenced by the title, deals with spaces which are generalizations of spaces of constant curvature. It is found that these spaces are those projective-plane spaces (in the sense of Weyl) for which the contracted curvature tensor B^h_{ijh} is symmetric. It is shown also that a space of this type can be immersed in a plane space of $n+1$ dimensions. This paper will appear in the TRANSACTIONS OF THIS SOCIETY.

14. Professor T. R. Hollcroft: *Self-dual space curves.*

In the transcript of a paper presented to the Society on May 2, 1925, which was concerned chiefly with self-dual plane curves, the final paragraph deals with self-dual space curves. In addition to this, the following have since been found: (1) singularity limits for self-dual space curves; (2) conditions on a plane curve that it be the projection of a self-dual space curve, and the condition on a space curve that its plane projection be self-dual; (3) conditions under which both the space curve and its plane projection curve are self-dual. In the last case it is found that on a quadric such curves exist only for the genus unity and the orders 8 to 21 inclusive. Similar limits are found for surfaces of higher order.

15. Professor C. R. Adams: *On the existence of solutions of a linear partial pure difference equation.*

The linear partial pure difference equation (1): $f(x+1, y+1)=a(x, y)f(x, y)$, in which $f(x, y)$ is to be determined, is carried by a simple linear transformation into the ordinary equation, containing a parameter, (2): $g(z+1, \zeta)=b(z, \zeta)\,g(z, \zeta)$. If $a(x, y)$ is a polynomial, $b(z, \zeta)$ will be likewise; let the latter be arranged according to descending powers of z, the leading term being $p_0(\zeta)z^\mu$. Equation (2) is satisfied formally by a series like that which satisfies the ordinary equation of first order without a parameter except that each constant (other than μ) is here replaced by a rational function of ζ. This series is in general divergent. There exist, however, two solutions analytic for z in the entire finite plane and ζ in any finite region S not containing a zero of $p_0(\zeta)$, and asymptotically represented, one in any left half, the other in any right half, of the z-plane, by the formal series, ζ being in S. A thorough investigation of the equation of nth order of type (1) will presuppose a knowledge of the character of the solutions of the ordinary equation of nth order in the case in which the characteristic equation has multiple, infinite, or zero roots. A study of this preliminary problem is now in progress by the author.

16. Professor E. L. Dodd: *The convergence of a general mean of measurements to the true value.*

This paper appears in full in the present issue of this BULLETIN.

17. Professor J. L. Walsh: *On the development of a function of a complex variable by means of polynomials.*

The principal results of this paper are the following. If the function $F(z)$ is continuous on the Jordan curve C in whose interior the point $z=0$ lies, then on C the function $F(z)$ is developable in a uniformly convergent series of polynomials in z and $1/z$. If the function $F(z)$ is continuous on an arc C' of a Jordan curve, then on C' the function $F(z)$ is developable in a uniformly convergent series of polynomials in z.

18. Professor Philip Franklin: *The fundamental theorem of almost periodic functions of two variables.*

In a paper presented at the October meeting, the author began the extension of the theory of almost periodic functions of one variable, of Harold Bohr, to the corresponding functions of two variables. In this paper the theory is extended, and in particular a set of theorems leading up to the analogue of Parseval's theorem for Fourier series, the fundamental theorem, are obtained.

19. Professor L. C. Mathewson: *A simple proof of a theorem of Kronecker's.*

Although others had implied the following theorem, Kronecker was the first to give an explicit proof of it:* *Every non-cyclic abelian group is the direct product of independent cyclic groups.* The simple proof given in the present paper is obtained by use of the following lemma, which itself is readily established by mathematical induction: If G is an abelian group of order p (a prime) and s is an element of highest period in G, then G is the direct product of the cyclic group S generated by s, and a largest subgroup in G having only the identity in common with S; in other words, G is the direct product of S and G/S.†

20. Professor R. L. Moore: *Conditions under which the sum of an infinite number of continua is itself a continuum.*

In this paper it is shown that if, in space of two dimensions, G is an infinite collection of mutually exclusive continua, and M, the sum of these continua, is closed and bounded, and K is a bounded continuum which has at least one point in common with each continuum of G, then in order that M should be a continuum it is necessary and sufficient that it should be impossible to express the common part of K and M as the sum of two

* BERLINER SITZUNGSBERICHTE, 1870, pp. 881–889.

† Cf. Miller, Blichfeldt, and Dickson, *Finite Groups*, 1916, p. 88.

mutually exclusive closed point sets T_1 and T_2 such that no continuum of G contains both a point of T_1 and a point of T_2. It is shown, by an example, that this theorem does not remain true on the removal of the condition that M be bounded.

21. Dr. H. M. Gehman: *Some conditions under which a continuum is a continuous curve.*

In this paper it is shown that a bounded continuum is a continuous curve if it can be expressed as the sum of a collection (C) of continuous curves, such that (1) at most a finite number are of diameter greater than any positive number, and (2) if (D) is any infinite collection of elements of (C), then some element of (C) has points in common with infinitely many of the elements of (D). A set of conditions of the same type are given under which every subcontinuum of the given continuum is also a continuous curve. A number of examples are given to show the independence of the various conditions.

22. Dr. H. M. Gehman: *Some relations between a continuous curve and its subsets.*

The author shows that if every proper subcontinuum of a continuum M is a continuous curve, then M is continuous curve. This shows that pseudo-Jordan curves, as defined by Yoneyama (TÔHOKU MATHEMATICAL JOURNAL, vol. 12 (1917), p. 157) cannot exist. The following is an extension of a theorem due to R. L. Wilder (this BULLETIN, vol. 29 (1923), p. 118, first paper, Theorem 2): If N is a continuous curve which is a proper subset of a continuous curve M, then $M-N$ can contain at most a finite number of maximal connected subsets of diameter greater than any given positive number.

23. Dr. H. M. Gehman: *Irreducible continuous curves about a set of points.*

It is first shown that the conditions that a set be *an irreducible continuous curve about a set of points* A, and that it be *a continuous curve which is an irreducible continuum about* A, are equivalent. The following theorems are then proved: (1) If A is a closed subset of a continuous curve S, then S contains an irreducible continuous curve about A, if and only if every maximal connected subset of A is a continuous curve, and not more than a finite number are of diameter greater than any positive number. (2) If M is a continuous curve, the set which is irreducible with respect to the property of being a closed set about which M is an irreducible continuous curve, consists of all end points of M, all points on simple closed curves in M, and all limit points of these two sets. (3) A continuum M contains an irreducible continuous curve about every closed disconnected subset of M if and only if every subcontinuum of M is a continuous curve. (4) If M is an irreducible continuous curve about a closed set A, every subcontinuum K of M is connected im kleinen at every point not in A, and if K contains every point of a maximal connected subset C of A, then K is also connected im kleinen at all points of C.

24. Dr. R. G. Lubben: *The separation of plane point sets by curves.*

The author shows that if K and H are closed and bounded point sets whose common part is a totally disconnected point set T, and K is a continuum, then a necessary and sufficient condition that there exist a simple closed curve containing a subset of T but no point of $K+J-T$, and separating $K-T$ from $H-T$, is the following: $H-T$ (1) is a subset of a complementary domain of K, and (2) is not separated by K near T. If it be assumed in addition that H is a continuum, then part (1) of the condition is superfluous. If K, H, and T are point sets, the statement "$H-KH$ is not separated by K near T" means that if P is a point of T and C is a circle about P, there exists a circle C_1 with the same center such that any pair of points x and y of $H-KH$ within C_1 can be joined by a connected point set which contains no points of K and is entirely within C.

25. Dr. R. G. Lubben: *Concerning connected plane point sets.*

The author shows that if T is a closed, totally disconnected subset of a bounded continuum K, S is the set of all points, and $S-K$ is not separated by K near T, then $K-T$ is connected. If G is a countable collection of mutually exclusive continua $g_1, g_2, g_3, \cdots$, and M, the sum of all these continua, is closed and bounded, and K is a bounded continuum containing at least one point in common with each continuum of the collection G, and for each i, h_i denotes the point set $g_i \cdot (K-K \cdot M)'$, and H_i is a continuum containing h_i, then if the point set $(K-K \cdot M+\Sigma_{i=1}^{\infty} H_i)'$ is bounded, it is a continuum. The theorem is not true if the elements of G are not mutually exclusive or if their number is not countable.

26. Mr. G. T. Whyburn: *Some theorems concerning domains and their boundaries.*

In this paper the following theorems are proved: (1) In order that a bounded domain D have the property S* it is necessary and sufficient that every point of the boundary of D be accessible from all sides from D in the sense of Schoenflies. (2) In order that the boundary M of a simply connected bounded domain be a continuous curve it is necessary and sufficient that every subcontinuum of M contain an uncountable set of points such that M is disconnected by the omission of any two of them. (3) The same condition as in (2) is sufficient (though not necessary) in order that any bounded continuum M be a continuous curve every subcontinuum of which is a continuous curve. (4) A point set K is not the common boundary of three mutually exclusive domains if it contains two

* R. L. Moore, Fundamenta Mathematicae, vol. 3, pp. 232–237.

points each of which is accessible from all three of these domains. (5) If a bounded and closed point set separates the plane in the weak sense but contains no proper subset which does so, then it is a simple closed curve. (6) If a domain is uniformly connected im kleinen, every point of its boundary is accessible.

27. Mr. C. M. Cleveland: *Concerning certain continua which contain no cut points in the weak sense.*

In this paper the author shows that if M is a bounded continuum which contains the interior of no circle and which has just two complementary domains, and, for each point P of M, $M-P$ is strongly connected, then M is a simple closed curve.

28. Mr. Orrin Frink: *A proof of Petersen's theorem.*

Petersen's theorem is a step towards proving the four-color theorem. It states that all regular graphs of the third degree with fewer than three leaves are colorable. The proofs given by Petersen, Brahana, and Errera are complicated and involve shrinking and counting processes. It is here shown that a simple proof by mathematical induction can be given. The theorem is first proved for graphs without leaves and is then proved for the general case.

29. Professor J. F. Ritt: *Meromorphic functions with addition or multiplication theorems.*

Let $f(z)$ represent a meromorphic function. It is desired to determine the circumstances under which a linear $\lambda(z)$ exists such that $f[\lambda(z)]$ is an algebraic function of $f(z)$. A first simplification of the problem results from a theorem of Picard, according to which, if two meromorphic functions are algebraically related to one another, the genus of the relation cannot exceed unity. It is proved that when $f[\lambda(z)]$ and $f(z)$ satisfy a relation of genus unity, they are both elliptic functions of a single integral function $g(z)$, where $g[\lambda(z)]$ is linear in $g(z)$. The rather lengthy statement of results for the case of genus zero will be found in the PARIS COMPTES RENDUS for January 18, 1926.

30. Professor J. F. Ritt: *Real functions with algebraic addition theorems.*

It is a well known theorem of Weierstrass that if an analytic function $\varphi(z)$ satisfies an algebraic relation (1): $G[\varphi(x), \varphi(y), \varphi(x+y)]=0$, $\varphi(x)$ is either an algebraic function, or an algebraic function of $e^{\mu x}$(μ constant), or an algebraic function of $\wp(x)$. If we restrict ourselves to the real domain, it is natural to require less than analyticity of $\varphi(x)$; for instance, mere continuity. In that case, $\varphi(x)$ does not have to be of one of the three types just mentioned. The present paper considers functions $\varphi(x)$ satisfying an irreducible relation (1) and continuous in an interval $(0, a>0)$. It is proved that $\varphi(x)$ is piecewise analytic in $(0, a)$, and that if $\varphi_1(x)$ is one of the analytic functions of which $\varphi(x)$ is composed, the remaining functions are all of the form $\varphi_1(x+a)$ with a constant. These analytic functions all belong to one of the three types met in Weierstrass's theorem.

31. Professor G. A. Pfeiffer: *On the decomposition of unbounded continua.*

It is proved in this paper that any continuum in the plane which is the sum of a countable number of mutually exclusive closed sets is decomposable, i.e., is the sum of two proper subsets, each of which is a continuum. Mazurkiewicz gave the first example of such a continuum which by a theorem of Sierpinski must be unbounded. An example of a somewhat different nature is also given by the author.

32. Dr. C. H. Langford: *Some theorems on deducibility.*

This paper has to do with the consequences entailed by sets of defining properties for three types of dense series: those without extreme elements, those with a first but no last element, and those with a first and a last element. These sets involve first-order functions only. A first-order function is a function whose values are first-order propositions, and a first-order proposition is a proposition which involves variables denoting individuals but does not involve any variable functions. We consider the class of all first-order functions which can be formulated on the base K, R_2. It is shown that any first-order function on this base has its truth-value determined by any one of these sets. This is a form of the problem of categoricalness. Each of the sets is categorical with respect to first-order functions.

33. Dr. H. W. Brinkmann: *On the condition that a Riemann n-space be immersible in* $(n+2)$*-dimensional euclidean space.*

In a previous paper the author has determined the condition that an n-dimensional Riemann space be immersible in euclidean $(n+1)$-space. The case under consideration in the present paper is more complicated and interesting from several points of view. The method used is that of projective geometry.

R. G. D. Richardson,
Secretary.