equation for one root we are then able to find 2 roots* of the $(t-1)$th resolvent and 2^t roots of the original equation. The procedure for finding the m remaining roots is obvious.

It is a fairly simple matter to write out formulas, by this method, for the roots of equations of lower degree than the sixth, but for higher degree equations the work becomes extremely complicated.

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THE CONDITIONS FOR A FIXED POINT IN PROJECTIVE DIFFERENTIAL GEOMETRY

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1. *Introduction.* In the projective differential geometry of Wilczynski, as applied to various special theories, a local frame of reference is found to be useful. When theorems† which involve fixed points are proved by Wilczynski's methods, the conditions satisfied by the coordinates of such points, referred to such local frames, are naturally of importance. It is a conspicuous fact that these conditions invariably involve the adjoint system‡ of differential equations. This fact the present paper undertakes to explain.

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* This requires the solution of odd degree equations only. See *On the solution of algebraic equations with rational coefficients*, AMERICAN MATHEMATICAL MONTHLY, June, 1924, p. 286.

† See A. F. Carpenter, *Some fundamental relations in the projective differential geometry of ruled surfaces*, ANNALI DI MATEMATICA, (3), vol. 26 (1917), pp. 285 et seq. Also A. L. Nelson, *Plane nets with equal invariants*, RENDICONTI DI PALERMO, vol. 41 (1916), pp. 251 et seq.

‡ More precisely, *geometric adjoint* system, in the language of Green (cf. *Memoir on the general theory of surfaces and rectilinear congruences*, TRANSACTIONS OF THIS SOCIETY, vol. 20 (1919), p. 106). This will be further discussed in §2.

The term "system of differential equations" will be understood in this paper to mean "completely integrable system of partial differential equations," and to include the system of one or more ordinary differential equations as a special case.

The basic systems of differential equations used in the various special theories differ among themselves in respect to (1) the number of independent variables; (2) the number of dependent variables; (3) the number of arbitrary constants. The results of this paper hold for n independent variables, n being a definitely positive integer. For the sake of definiteness and convenience of notation, only the case of one dependent variable and four arbitrary constants will be discussed. The modifications necessary for the general situation are obvious.

2. *Geometric Adjoint Systems.* In a given special theory, the geometric locus under consideration may be thought of as originally a point locus, the homogeneous coordinates of the generating point, P_y, being any four linearly independent solutions of the basic system of differential equations. This basic system will be denoted by (y). But, dually, the locus may also be considered as a plane locus, and it becomes necessary to determine a system of the same type as (y) satisfied by the plane coordinates of the plane which corresponds to P_y. Now neither the point coordinates of P_y nor the plane coordinates of the corresponding plane are unique. This arbitrariness in the coordinates (y_1, y_2, y_3, y_4) of P_y is reflected in the fact that corresponding to (y) there are ∞^1 systems, all equivalent under the transformation $y=\lambda\bar{y}$ (where λ is a function of the independent variables), all of which correspond to the same point P_y. A similar statement holds for the system dual to (y). Thus, in a general sense, we might (and will in this paper) apply the term geometric adjoint to any member of the ∞^1 systems satisfied by the ∞^1 sets of coordinates of the plane dual to P_y.

If, however, we use only those coordinates of the dual plane which are absolutely cogredient with the point coordinates of P_y under the transformation $y=\lambda\bar{y}$, then, corresponding to a given system (y) there is only one geometric adjoint. It is in this restricted sense that Wilczynski* uses the term. It should be noted, however, that in this restricted sense, the Lagrange

* *Projective differential geometry of curves and ruled surfaces*, Leipzig, B. G. Teubner, 1906, p. 138.

adjoint* of a linear homogeneous ordinary differential equation is not a geometric adjoint.

3. *The Conditions for a Fixed Point.* Consider the points P_y, P_z, P_ρ, P_σ, whose coordinates, with reference to some fixed tetrahedron T_1, are respectively (y_1, y_2, y_3, y_4), (z_1, z_2, z_3, z_4), $(\rho_1, \rho_2, \rho_3, \rho_4)$, $(\sigma_1, \sigma_2, \sigma_3, \sigma_4)$, where z_i, ρ_i, σ_i are such linear homogeneous functions of y_i and its derivatives $(i=1, 2, 3, 4)$ that the tetrahedron T_2 formed by these four points is non-degenerate. That is, we assume that, for certain ranges of values of the independent variables, the determinant

$$\Delta = \begin{vmatrix} y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & x_4 \\ \rho_1 & \rho_2 & \rho_3 & \rho_4 \\ \sigma_1 & \sigma_2 & \sigma_3 & \sigma_4 \end{vmatrix}$$

does not vanish.

The functions

$$\theta_i = \alpha y_i + \beta z_i + \gamma \rho_i + \delta \sigma_i \ , \qquad (i=1,2,3,4) \ , \tag{1}$$

where α, β, γ, δ are arbitrary functions of the independent variables, are the coordinates (with reference to T_1) of an arbitrary point P_θ. With reference to T_2, the coordinates of P_θ may be taken as $(\alpha, \beta, \gamma, \delta)$. Solving equations (1) for these new coordinates, we have

$$\begin{aligned} \Delta \cdot \alpha &= \theta_1 Y_1 + \theta_2 Y_2 + \theta_3 Y_3 + \theta_4 Y_4 \ , \\ \Delta \cdot \beta &= \theta_1 Z_1 + \theta_2 Z_2 + \theta_3 Z_3 + \theta_4 Z_4 \ , \\ \Delta \cdot \gamma &= \theta_1 P_1 + \theta_2 P_2 + \theta_3 P_3 + \theta_4 P_4 \ , \\ \Delta \cdot \delta &= \theta_1 \Sigma_1 + \theta_2 \Sigma_2 + \theta_3 \Sigma_3 + \theta_4 \Sigma_4 \ , \end{aligned} \tag{2}$$

where Y_i, Z_i, P_i, Σ_i are the cofactors of y_i, z_i, ρ_i, σ_i, in Δ. These equations show that the new coordinates of P_i may be taken as

$$(\Sigma\theta_i Y_i \ , \ \Sigma\theta_i Z_i \ , \ \Sigma\theta_i P_i \ , \ \Sigma\theta_i \Sigma_i) \ . \tag{3}$$

Suppose now that P_θ is fixed. Then θ_1, θ_2, θ_3, θ_4, its coordi-

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* Cf. Wilczynski, loc. cit., pp. 40 et seq.

nates with reference to T_1, may without loss of generality* be assumed to be constants, and (for example) the first of the new coordinates (3), by the theory of differential equations, must be a solution of any completely integrable system of linear homogeneous partial differential equations satisfied by $Y_i (i=1, 2, 3, 4)$. In particular, if (Y) denotes that system satisfied by the Y's which is of the same type as (y), and if a similar notation holds for the other letters, then we have proved the

THEOREM. *If P_θ is fixed, then by a transformation of the type $y=\lambda\bar{y}$, the new coordinates of P_θ, with reference to the local tetrahedron T_2, may be made to satisfy the systems (Y), (Z), (P), (Σ), respectively.*

Conversely, any solutions of (Y), (Z), (P), (Σ) are of type (3), where the θ's are constants, and the Y's, Z's, P's, Σ's are linearly independent solutions of (Y), (Z), (P), (Σ), respectively. Setting up equations similar to (2), and solving, we obtain (1), which show that α, β, γ, δ may be taken as the coordinates, with reference to T_2, of a fixed point.

The connection of the foregoing with the geometric adjoint is obvious. (Y_1, Y_2, Y_3, Y_4), (Z_1, Z_2, Z_3, Z_4), (P_1, P_2, P_3, P_4), $(\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4)$ are plane coordinates of the faces of the tetrahedron T_2. In special theories, T_2 is naturally taken in such a way as to make one of its faces, say $(\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4)$, the dual of P_y, so that the system (Σ) is a geometric adjoint (in the unrestricted sense) of (y). Consequently we must expect to find that the coordinate δ of the fixed point is a solution of a geometric adjoint of the original basic system.

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* A transformation of the type $\theta=\lambda\bar{\theta}$ will make them constants.