NÖRLUND ON CALCULUS OF DIFFERENCES

Vorlesungen über Differenzenrechnung. By Niels Erik Nörlund. Berlin, Springer, 1924. ix + 551 pages.

This is the first book to develop the theory of the difference calculus from the function-theoretic point of view and to include a significant part of the recent researches having to do with the analytic and asymptotic character of the solutions of linear difference equations. As such it is an important contribution to the mathematical literature and will render a service not procurable from any of its predecessors. The author presents a connected account of what appears to him to be the most important and the best developed domains of the difference calculus. The book is intended to give a preliminary view of the field and to facilitate the reading of the original memoirs. With this purpose in view the author sometimes omits proofs (too frequently, we think) leaving the reader to find them in the indicated memoirs. There is unfortunately little attempt to make clear the historical aspects of the subject, with the result that the casual reader of the book will get an erroneous impression of the actual historical development. It is the reviewer's opinion that this danger is not sufficiently safeguarded by the reference to Nörlund's *Neuere Untersuchungen über Differenzengleichungen* in the ENZYKLOPÄDIE DER MATHEMATISCHEN WISSENSCHAFTEN, Band II C 7.

The most important earlier books on the difference calculus are those of Lacroix, Boole, Markoff, Pascal, Pincherle, and Wallenberg and Guldberg. They proceed principally from the algebraic standpoint (with a partial exception in the case of the last) and often employ symbolic methods. In Nörlund's book the point of view is function-theoretic throughout. This marks well the shift of interest on the part of those who have developed the difference calculus. The more formal aspects, elementary solutions, interpolations and such things engaged the attention of the founders, among whom were Newton, Taylor, Stirling, Laplace and Gauss. "Dann kam eine Zeit," as Nörlund says (in his introduction, page 2), "in der das Gebiet fast ganz brach lag und keine nennenswerten Fortschritte erzielt wurden. Heute jedoch zeigt sich wieder frisches Leben; unter neuen Gesichtspunkten und mit besseren Hilfsmitteln als früher hat man das Studium der zum Teil bis in die Anfänge der Differenzrechnung zurückreichenden Probleme wieder aufgenommen, und in den letzten Jahren ist eine ziemlich umfangreiche Literatur emporgewachsen, aus der als besonders wertvoll einige von Pincherle, Birkhoff, Carmichael, Galbrun und Perron herrührende Arbeiten zu nennen sind."

To this list of recent investigators the name of Nörlund himself is to be added and is to be given a place of large importance. We believe that the name of Poincaré should also be mentioned here as one who made an important contribution to the difference calculus in 1885 and thus gave the first intimation of the possible rejuvenescence of the theory of difference equations. Nörlund himself (p. 272) refers to this work of Poincaré as opening up for the first time a real theory of linear difference equations by means of a (now celebrated) theorem on the asymptotic behavior of the solutions of such equations.

The main part of Nörlund's text covers 453 pages. About one-third of the whole (making up the larger part of the first 200 pages) is given to the problem of finite integration and its congeners and generalizations. By a finite integral of a function $\varphi(x)$ we mean a function $f(x)$ satisfying the equation $[f(x+\omega)-f(x)]/\omega = \varphi(x)$. A related problem is that of determining a function $g(x)$ which satisfies the equation $[g(x+\omega)+g(x)]/2 = \varphi(x)$. In each of these equations ω is an arbitrary complex number. For many purposes ω may be given the value unity without loss of generality. But, for the theory of finite integration, as developed by Nörlund, it plays the role of an independent variable. The generalization of finite integration in the direction of what may be called repeated finite integration and a similar generalization of the related problem associated with the second of the foregoing equations are treated at length. As an example of the first we have the solution of the second order equation

$$\frac{f(x+\omega_1+\omega_2)-f(x+\omega_1)-f(x+\omega_2)+f(x)}{\omega_1\,\omega_2} = \varphi(x)$$

for $f(x)$ in terms of the given function $\varphi(x)$. Special cases of these problems are also treated at length; and, in particular, the determination of the finite integral of $\nu x^{\nu-1}$ when $\omega = 1$, together with the solution of related special problems. An elegant treatment is given of the theory of the usual and the higher Bernoulli and Euler polynomials.

The chief attention of the first seven chapters (except for a treatment of gamma functions and related functions in the twenty-one pages of Chapter V) is given to the various aspects of the problem of finite integration, or, as the author names it, that of finding the "sum" of a given function. The general finite integral of the given function $\varphi(x)$ involves an arbitrary periodic function of x of period ω. One principal purpose of Nörlund is to develop a direct algorithm by means of which finite integrals having characteristic properties can be selected from the totality of finite integrals. This purpose he achieves by means of appropriate definitions of summability applied to the direct formal sums

$$-\omega\sum_{s=0}^{\infty}\varphi(x+s\omega), \qquad \omega\sum_{s=1}^{\infty}\varphi(x-s\omega)$$

as formal finite integrals of the given function $\varphi(x)$. By this means he is able to set up appropriate definitions of finite integrals of such sort as to characterize them in a very satisfactory way and to render them determinate except for an additive constant. Thus in this book and in the relevant parts of his preceding memoirs the author brings the problem of finite integration to a stage of development comparable to that of integration in the infinitesimal calculus. Here his treatment has much of novelty and importance. One can only wish that he had more effectively brought out the relation of his own methods and results to those of other (and especially earlier) investigators who have had occasion to treat finite integration either for its own sake or as incidental to some larger problem in which it occurs.

Chapter eight (with 58 pages) is given to various types of interpolation series, with special attention to those of Stirling and Newton and to the character of functions defined as sums of such series.

An all-too-short development of the theory of factorial series is given in the ninth chapter, attention being confined to the most important results and especially to those which will be needed in later chapters.

Chapters ten to thirteen inclusive (pages 272–386) are devoted to the general theory of linear homogeneous difference equations. The first one opens with a general existence theorem developed by a method due to Nörlund. Then comes Ostrowski's recent proof of Hölder's theorem to the effect that the gamma function can not satisfy an algebraic differential equation, thus bringing out the fact that linear difference equations introduce us to a class of transcendental functions not brought to light by the study of differential equations. The generalizations or extensions of Hölder's theorem are not treated. A few elementary matters then follow and the chapter closes with a final section (of 13 pages) on Poincaré's theorem concerning the asymptotic character of solutions of difference equations.

The eleventh chapter (pages 314–352) is devoted to homogeneous linear difference equations with rational coefficients. It gives a rather extended development of a considerable part of the beautiful theory of these equations. The leading reference (page 314) is to Nörlund's 1915 paper in the ACTA MATHEMATICA and the methods employed in the chapter are due principally to him. In another and later part of the volume, namely on page 379 in the thirteenth chapter, is the statement that Birkhoff obtained (for these equations) many interesting and beautiful results "zu denen wir in Kapitel 11 unter anderen Gesichtspunkten gelangt sind." When it is remembered that Birkhoff's work was published in 1911, four years before that of Nörlund, it is seen to be desirable that Chapter eleven should contain a brief analysis of the relation between the work of the two authors with a clear indication of priority, especially if it is true (as I believe it to be) that the most fundamental known properties of these equations are first developed in Birkhoff's memoir.

The twelfth chapter (pages 353-378) contains a development of the theory of those linear homogeneous difference equations whose coefficients can be expressed by means of factorial series. It contains an important contribution due to Nörlund himself. The method is one of the most pleasing yet known for treating the theory of linear difference equations; and I expect to see it developed by generalization so as to be still more effective. Factorial series afford one of the most important tools of analysis; and I believe that both they and certain of their generalizations have an importance beyond that which is generally recognized.

The treatment of linear homogeneous difference equations is brought to a close in the thirteenth chapter (pages 379-386) with a very short and imperfect account of the researches of Birkhoff in 1911. With 107 pages devoted to other contributions to the theory of homogeneous linear difference equations, it is not satisfying to see this important memoir passed over so briefly. If the material contained in these 107 pages requires that space for its proper presentation, then, it appears to me that another hundred pages is needed to present with proper distribution of emphasis the matter which is omitted. This would be taken mostly from the work of three out of the five recent investigators singled out by Nörlund for special mention in the passage which we have quoted from his introduction. In the absence of such a modified distribution of emphasis the exposition fails to give a correct impression of the historical order of development of the subject and of the relative importance of the results so far attained.

In the fourteenth chapter (pages 387-414) we have a treatment of the non-homogeneous linear difference equation. Section one contains an elementary account of Lagrange's method of variation of parameters. Equations with constant coefficients are treated in Section two. An account of the researches of E. Hilb in 1922 is given in the final Section three. It appears to me that the most important memoir on the solutions of non-homogeneous linear difference equations is that of K. P. Williams (TRANSACTIONS OF THIS SOCIETY, vol. 14 (1913), pp. 209-240). Nörlund nowhere in his book utilizes the results of this memoir by Williams, and there is indeed no reference to Williams in the index though his papers are listed in the bibliography. The failure to utilize Williams's researches leaves the chapter inadequate to represent the actual state of the theory of non-homogeneous equations.

The last chapter (pages 415-455) is devoted to reciprocal differences and continued fractions. It is followed by a few pages of tables, an extensive bibliography and an index. In the bibliography the author has included all works known to him and dealing with the general theory of difference equations. From the very extensive special literatures of Bernoulli polynomials and gamma functions he has listed only the most important works. The bibliography is a very useful one. It is hardly to be expected that it should be complete. In fact I have found a considerable number of omissions by checking it against the partial bibliography which I have collected in an incidental way during the past fifteen years.

It is natural to expect that an exposition of a general subject should involve an important element depending on the personal interests of the author; and this is particularly true in the case of a book which is essentially the first in its field. But in the present book this element appears to me to have played too large a role in determining the distribution of emphasis and the selection of material. Much of the work in the first two hundred pages might well have been given with less fullness and the space so gained have been utilized in the presentation of some of the important matters which are omitted.

While this book will probably stand for some time as the best book in its field, and as such is therefore of great importance, it can not be regarded as having come near to being a definitive treatise on the difference calculus, even in its present state of development. Whatever one may think of the distribution of emphasis and selection of material in this volume there is still a definite need for another book with a quite different distribution and selection — one in which the personal equation of the author does not play so large a role.

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ENRIQUES ON ALGEBRAIC GEOMETRY

Lezioni sulla Teoria Geometrica delle Equazioni e delle Funzioni Algebriche. Vol. I and vol. II. By F. Enriques. Bologna, O. Chisini, 1915, 1918.

The reviewer, far from being a specialist in algebraic geometry, commences this short review with the misgiving that in the last two years of reading "at" the work of this famous author he has set his wisdom teeth into a sticky mouthful. It may however be said at the outset that the paucity, however unfair, of references to living workers in American universities,—a general reference to Osgood's *Funktionentheorie* and particular ones to Scott (C. A. Scott) and "Angas" ("Ch. Angas Scott"(!)),—indicates that the field is one in which few of us are expert, and therefore not only that the point of view of the reviewer will be that of most of his readers, but also that the treatise itself unbars a field which some of us might well explore. The reviewer tried the experiment of lecturing from it to capable, and patient, advanced students during the past academic year.