A HISTORICAL NOTE ON GIBBS' PHENOMENON IN FOURIER'S SERIES AND INTEGRALS

BY H. S. CARSLAW

In 1899, Gibbs called attention* to the fact that for large values of n the approximation curves

$$y = S_n(x) = 2\sum_1^n (-1)^{r-1} \frac{\sin rx}{r},$$

for the Fourier's series which represents $f(x) = x$ in the interval $-\pi < x < \pi$, fall from the point $(-\pi, 0)$ at a steep gradient to a point very nearly at a depth $2\int_0^\pi [(\sin\alpha)/\alpha]\,d\alpha$ below the axis of x, then oscillate above and below $y = x$ close to this line until x approaches π, when they rise to a point very nearly at a height $2\int_0^\pi [(\sin\alpha)/\alpha]\,d\alpha$ above the axis, and then fall rapidly to $(\pi, 0)$.

At the point of discontinuity, where $x = \pi$, in the series $2\sum_1^\infty (-1)^{r-1}(\sin rx)/r$ the approximation curves thus tend to coincide, not with the segment joining the points (π, π) and $(\pi, -\pi)$, but with the straight line whose ends are the points

$$\left(\pi, \pi + \frac{D}{\pi}\int_\pi^\infty \frac{\sin\alpha}{\alpha}\,d\alpha\right)$$

and

$$\left(\pi, -\pi - \frac{D}{\pi}\int_\pi^\infty \frac{\sin\alpha}{\alpha}\,d\alpha\right),$$

where $D = f(\pi + 0) - f(\pi - 0)$, the amount of the "jump" in the sum of the series at that point.

In 1906, Bôcher showed† that the same phenomenon occurred in general in the Fourier's Series for the arbitrary

* NATURE, vol. 59 (1899), p. 606.

† ANNALS OF MATHEMATICS, (2), vol. 7, (1906).

function $f(x)$ in the neighborhood of a point of discontinuity whose abscissa is a, when $f(a \pm 0)$ exist. The approximation curves approach a limiting curve in which the vertical portion is the line joining the points $(a, f(a-0))$ and $(a, f(a+0))$; *but this vertical line has to be produced beyond these points by an amount that bears a definite ratio to the magnitude of the jump.*

Bôcher gave the name *Gibbs' phenomenon* to this property of the approximation curves of the Fourier's series. Both Bôcher and Gibbs were under the impression that the property had remained unsuspected till Gibbs discovered it for the series

$$2\left(\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \cdots\right)$$

in 1899.

In my book on *Fourier's Series and Integrals,** I repeat this statement, and in the review of the third edition of Picard's *Traité d'Analyse* by C. N. Moore,† it is again to be found in print. *However, it is now known that Gibbs's phenomenon had been observed about* 50 *years earlier than* 1899. Indeed, in 1848, Wilbraham,‡ a B. A. of Trinity College, Cambridge, discussed the equation

$$y = \cos x - \frac{1}{3}\cos 3x + \frac{1}{5}\cos 5x - \cdots.$$

His object was to show that this equation represents "a locus composed of separate straight lines, of which each is equal to π, parallel to the axis of x, and at a distance $\pm \frac{1}{4}\pi$ alternately above and below it, joined by perpendiculars which are themselves part of the locus" but extend beyond the ends $\pm \frac{1}{4}\pi$ by a distance $\frac{1}{2}\left|\int_\pi^\infty [(\sin \alpha)/\alpha]d\alpha\right|$ at either extremity.

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* Carslaw, *Fourier's Series and Integrals,* 2d ed., 1921, p. 266.

† This Bulletin, vol. 30 (1924), p. 555.

‡ *On a certain periodic function*, Cambridge and Dublin Mathematical Journal, vol. 3 (1848), p. 198.

It could not be expected that Wilbraham in 1848 would have a clear idea of what is meant by the sum of the series for the value x_0: i. e., in our notation $\lim_{n \to \infty} S_n(x_0)$. In dealing with this sum he allows x to be a function of n which tends to x_0 as $n \to \infty$; he takes the sum $S_n(x)$ for the positive integer n and the corresponding value of x_0 and he then lets n tend to infinity. But, as his diagrams show, he saw quite clearly the behavior of the approximation curves of his series of cosines, and he remarked that "a similar investigation of the equation

$$y = \sin x - \frac{1}{2} \sin 2x + \frac{1}{3} \sin 3x - \cdots$$

would lead to an analogous result".

This second series, it will be seen, is the same as that which Gibbs used in 1899.

Wilbraham's paper seems to have attracted little attention and to have been forgotten until Burkhardt quoted it in his article on *Trigonometrische Reihen und Integrale* (*bis etwa* 1850).* The fact that Gibbs' phenomenon had been observed earlier is mentioned also by Hilb and Riesz in their recently published article *Neuere Untersuchungen über trigonometrische Reihen.*†

We may still call this property of Fourier's series (and certain other series) Gibbs' phenomenon; but we must no longer claim that the property was first discovered by Gibbs.

In the review of Picard's *Traité d'Analyse* mentioned above, Moore comments on the title which Picard gives to the section in which Gibbs' phenomenon is discussed, namely, "*Phénomène de Du Bois-Reymond et Gibbs*". He concludes that it is wrong to associate Du Bois-Reymond's name with this discovery. In this I think the reviewer is right. But it is certainly true that, if Du Bois-Reymond

* ENCYKLOPÄDIE DER MATHEMATISCHEN WISSENSCHAFTEN, Bd. II, Teil I, 2. Hälfte, p. 1049, 1914.

† ENCYKLOPÄDIE, Bd. II, Teil 3, p. 1203, 1924.

had not made a curious slip as to the values which the integral $\int_0^{nx}[(\sin\alpha)/\alpha]\,d\alpha$ can take when $n\rightarrow\infty$ and $x\rightarrow 0$ simultaneously, he would have enunciated Gibbs' phenomenon both for Fourier's series and Fourier's integral in 1874.

The memoir* to which Picard refers is well known. In it he distinguishes carefully between

$$\lim_{\varepsilon\rightarrow 0}\lim_{n\rightarrow\infty} f(x\pm\varepsilon, n),$$

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and the case when n tends to infinity and ε to zero simultaneously: e. g. $\varepsilon=\varphi_1(t)$ and $1/n=\varphi_2(t)$, while as t tends to its limit, $\varphi_1(t)$ and $\varphi_2(t)$ tend to zero.

He uses the symbol Lim for such an arbitrary simultaneous approach of n to infinity and ε to zero, and his work is based on the second theorem of mean value. But even when his results are correct, the argument is not always clear or rigorous.

Yet he finds (loc. cit. § 9) that in the case of Fourier's integral, when $x\rightarrow x_1$ and $h\rightarrow\infty$, we have

$$\begin{aligned}\operatorname{Lim}\frac{1}{\pi}\int_0^h d\alpha\int_{-\infty}^{\infty} d\beta\, f(\beta)\cos\alpha(\beta-x)\\ =\frac{1}{2}\{f(x_1+0)+f(x_1-0)\}\\ +\left\{\frac{f(x_1+0)-f(x_1-0)}{\pi}\right\}\operatorname{Lim}\int_0^{h(x-x_1)}\frac{\sin\alpha}{\alpha}\,d\alpha,\end{aligned}$$

while for Fourier's series, when $x\rightarrow x_1$, and $n\rightarrow\infty$, we have

$$\begin{aligned}\operatorname{Lim} S_n(x) = {}^1/_2\,\{f(x_1+0)+f(x_1-0)\}\\ +\left\{\frac{f(x_1+0)-f(x_1-0)}{\pi}\right\}\operatorname{Lim}\int_0^{n(x-x_1)}\frac{\sin\alpha}{\alpha}\,d\alpha.\end{aligned}$$

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* *Über die sprungweisen Werthänderungen analytischer Functionen*, MATHEMATISCHE ANNALEN, vol. 7 (1874), p. 241.

But he goes on to say that the integrals

$$\int_0^{h(x-x_1)} \frac{\sin\alpha}{\alpha}\,d\alpha \text{ and } \int_0^{n(x-x_1)} \frac{\sin\alpha}{\alpha}\,d\alpha,$$

which occur in these results, can take every value between $\pm\frac{1}{2}\pi$ — of course this should have been $\pm\int_0^\pi[(\sin\alpha)/\alpha]d\alpha$ — according to the way in which h and $(x-x_1)$, or n and $(x-x_1)$, tend to infinity and zero respectively. He thus concludes that the Fourier's integral and series at a point of ordinary discontinuity, in this simultaneous limit, represent all the values between $f(x_1-0)$ and $f(x_1+0)$, instead of extending beyond $f(x_1\pm 0)$ by the amount

$$\left|\left\{\frac{f(x_1+0)-f(x_1-0)}{\pi}\right\}\int_\pi^\infty \frac{\sin\alpha}{\alpha}\,d\alpha\right|.$$

Further, Du Bois-Reymond seems to have been the first to bring up the question of the behavior of Fourier's integral in this connection. And with a corresponding emendation his results are correct.

Early in 1925, I sent to the QUARTERLY JOURNAL OF MATHEMATICS a paper on Gibbs' phenomenon in one of the early sections of which I refer to the work of Wilbraham and Du Bois-Reymond. But the mistake about the discovery of Gibbs' phenomenon is so common that it seems proper to make this correction also in this BULLETIN.

SYDNEY, AUSTRALIA