THE TENSOR CHARACTER OF THE GENERALIZED KRONECKER SYMBOL*

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1. *Introduction.* In a previous paper† we have considered the use of the generalized Kronecker symbol $\delta^{r_1 r_2 \cdots r_m}_{s_1 s_2 \cdots s_m}$ in presenting the theory of determinants and we now proceed to show that it is an arithmetic tensor of the type indicated by its subscripts and superscripts, i. e., it is covariant of rank m and contravariant of rank m. By the statement that a tensor is arithmetic, we mean that its presentation is independent of the particular coordinate system in use and that it has the same numerical values for its various components at all points of space.‡

The generalized Kronecker symbol may be defined by means of the equation

$$(1) \qquad \delta^{r_1 r_2 \cdots r_m}_{s_1 s_2 \cdots s_m} = \begin{vmatrix} \delta^{r_1}_{s_1} & \cdots & \delta^{r_1}_{s_m} \\ \delta^{r_2}_{s_1} & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ \delta^{r_m}_{s_1} & \cdots & \delta^{r_m}_{s_m} \end{vmatrix}.$$

Here the labels r and s run independently over a set of n numbers $1, 2, \cdots, n$ and $\delta^r_s = 1$ if $r = s$, and $= 0$ if $r \neq s$; δ^r_s is the ordinary Kronecker symbol and it is usually denoted by g^r_s in the theory of relativity. It is there derived as the scalar product of the metric tensor g_{rs} and its reciprocal g^{rs}, but this mode of presentation is somewhat

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† *The Generalized Kronecker symbol and its application to the theory of determinants,* AMERICAN MATHEMATICAL MONTHLY vol. 32 (1925), p. 233. This paper will be denoted by the symbol (A) in references below.

‡ Cf. P. FRANKLIN, PHILOSOPHICAL MAGAZINE, (6), vol. 45 (1923), p. 998.

unfortunate since the tensor character of the symbol has nothing to do with the metric properties of the space.

The simplest procedure in proving the tensor character of the symbol $\delta^{r_1 r_2 \cdots r_m}_{s_1 s_2 \cdots s_m}$ is first to prove the theorem when $m = n$, the number of dimensions of the space. To do this we shall define a tensor of rank $2n$, contravariant of rank n and covariant of rank n, by the statement that its presentation in a particular coordinate system $(x^1, x^2, x^3, \cdots, x^n)$ is furnished by the values of the generalized Kronecker symbol $\delta^{r_1 r_2 \cdots r_n}_{s_1 s_2 \cdots s_n}$. Denoting the presentation of this tensor in any other coordinate system $(y^1, y^2, \cdots, y^n)$ by $\varepsilon^{r_1 r_2 \cdots r_n}_{s_1 s_2 \cdots s_n}$, we have to show that

$$\varepsilon^{r_1 r_2 \cdots r_n}_{s_1 s_2 \cdots s_n} = \delta^{r_1 r_2 \cdots r_n}_{s_1 s_2 \cdots s_n}$$

where the labels $(r_1, r_2, \cdots, r_n)$ and $(s_1, s_2, \cdots, s_n)$ may each be assigned independently any one of the values $(1, 2, \cdots, n)$. We have, from the definition of a tensor,

$$\varepsilon^{r_1 r_2 \cdots r_n}_{s_1 s_2 \cdots s_n} = \delta^{\alpha_1 \alpha_2 \cdots \alpha_n}_{\beta_1 \beta_2 \cdots \beta_n} \frac{\partial y^{r_1}}{\partial x^{\alpha_1}} \cdots \frac{\partial y^{r_n}}{\partial x^{\alpha_n}} \frac{\partial x^{\beta_1}}{\partial y^{s_1}} \cdots \frac{\partial x^{\beta_n}}{\partial y^{s_n}}$$

where we have adopted the convention of the previous paper (A), according to which *Greek* labels occurring twice in any term indicate summations. Writing*

$$(2) \qquad \delta^{\alpha_1 \alpha_2 \cdots \alpha_n}_{\beta_1 \beta_2 \cdots \beta_n} = \frac{1}{n!} \delta^{\alpha_1 \alpha_2 \cdots \alpha_n}_{\lambda_1 \lambda_2 \cdots \lambda_n} \delta^{\lambda_1 \lambda_2 \cdots \lambda_n}_{\beta_1 \beta_2 \cdots \beta_n}$$

we have

$$\varepsilon^{r_1 r_2 \cdots r_n}_{s_1 s_2 \cdots s_n}$$

$$= \frac{1}{n!}\left(\delta^{\alpha_1 \alpha_2 \cdots \alpha_n}_{\lambda_1 \lambda_2 \cdots \lambda_n} \frac{\partial y^{r_1}}{\partial x^{\alpha_1}} \cdots \frac{\partial y^{r_n}}{\partial x^{\alpha_n}}\right)\left(\delta^{\lambda_1 \lambda_2 \cdots \lambda_n}_{\beta_1 \beta_2 \cdots \beta_n} \frac{\partial x^{\beta_1}}{\partial y^{s_1}} \cdots \frac{\partial x^{\beta_n}}{\partial y^{s_n}}\right).$$

Now the expression on the right hand side is a summation of products of two factors of the type

$$\delta^{\alpha_1 \alpha_2 \cdots \alpha_n}_{l_1 l_2 \cdots l_n} \frac{\partial y^{r_1}}{\partial x^{\alpha_1}} \cdots \frac{\partial y^{r_n}}{\partial x^{\alpha_n}} \qquad \text{and} \qquad \delta^{l_1 l_2 \cdots l_n}_{\beta_1 \beta_2 \cdots \beta_n} \frac{\partial x^{\beta_1}}{\partial y^{s_1}} \cdots \frac{\partial x^{\beta_n}}{\partial y^{s_n}}$$

* See paper (A), equation $(2 \cdot 3)$.

respectively. Each of these factors is a determinant of order n (see $(3 \cdot 4)$, paper (A)) and since an inversion of any two of the labels $(l_1, l_2, \cdots, l_n)$ merely changes the sign of both factors we may write

$$\varepsilon^{r_1 r_2 \cdots r_n}_{s_1 s_2 \cdots s_n}$$
$$= \delta^{\alpha_1 \alpha_2 \cdots \alpha_n}_{1\ 2\ \cdots\ n} \frac{\partial y^{r_1}}{\partial x^{\alpha_1}} \cdots \frac{\partial y^{r_n}}{\partial x^{\alpha_n}} \cdot \delta^{1\ 2\ \cdots\ n}_{\beta_1 \beta_2 \cdots \beta_n} \frac{\partial x^{\beta_1}}{\partial y^{s_1}} \cdots \frac{\partial x^{\beta_n}}{\partial y^{s_n}}.$$

The product of the two determinants on the right is a determinant of which the element in the pth row and the qth column is

$$\frac{\partial y^{r_p}}{\partial x^{\sigma}} \cdot \frac{\partial x^{\sigma}}{\partial y^{s_q}} \quad \text{or} \quad \delta^{r_p}_{s_q}. \tag{3}$$

Hence

$$\varepsilon^{r_1 r_2 \cdots r_n}_{s_1 s_2 \cdots s_n} = \delta^{r_1 r_2 \cdots r_n}_{s_1 s_2 \cdots s_n}$$

which proves the theorem stated.

The tensor character of the symbols for the cases $m < n$ follows at once by contraction of the tensor $\delta^{r_1 r_2 \cdots r_n}_{s_1 s_2 \cdots s_n}$. Thus it is immediately apparent that

$$\delta^{r_1 r_2 \cdots r_{n-1}}_{s_1 s_2 \cdots s_{n-1}} = \delta^{r_1 r_2 \cdots r_{n-1} \alpha}_{s_1 s_2 \cdots s_{n-1} \alpha} \tag{4}$$

for in the summation on the right all the terms vanish unless $(r_1, r_2, \cdots, r_{n-1})$ are all different and $(s_1, s_2, \cdots, s_{n-1})$ all different and also α different from any of the $(r_1, r_2, \cdots, r_{n-1})$ and the $(s_1, s_2, \cdots, s_{n-1})$. The only term which has a value different from zero occurs, therefore, when $(r_1, r_2, \cdots, r_{n-1})$ and $(s_1, s_2, \cdots, s_{n-1})$ are arrangements of the same group of $n-1$ out of the n numbers $(1, 2, \cdots, n)$ and α is the remaining number. The equation (4) shows that $\delta^{r_1 r_2 \cdots r_{n-1}}_{s_1 s_2 \cdots s_{n-1}}$ is an arithmetic mixed tensor contravariant of rank $n-1$ and covariant of rank $n-1$. Proceeding similarly we arrive at the simple Kronecker symbol $\delta^r_s = \delta^{r \alpha_1 \alpha_2 \cdots \alpha_{n-1}}_{s \alpha_1 \alpha_2 \cdots \alpha_{n-1}}$ which is an arithmetic mixed tensor of the second rank.

It may not be superfluous to call attention again to the fact that these tensors are non-metric. The space for which they are defined is the general space of analysis situs, or topology, in which a point is merely a set of n ordered numbers. The usual presentation of, and notation for, the mixed tensor δ^r_s is therefore unfortunate. This starts with a symmetric covariant tensor g_{rs} of the second rank which furnishes the metric ground form $(ds)^2 = g_{\alpha\beta}dx^\alpha dx^\beta$ of a Riemann space. From this is derived a contravariant presentation g^{rs} of the metric tensor and the inner product $g^r_s = g^{r\alpha}g_{\alpha s}$ furnishes the mixed arithmetic tensor which we have denoted by δ^r_s.

2. *The Outer Multiplication of Tensors.* If we have two covariant tensors $a_{r_1r_2\cdots r_p}$ and $b_{s_1s_2\cdots s_q}$ of ranks p and q respectively we may derive from them, by means of the generalized Kronecker tensor, an alternating covariant tensor of rank $p+q$ as follows:

$$c_{r_1r_2\cdots r_ps_1\cdots s_q} = \delta^{\alpha_1\cdots\alpha_p\beta_1\cdots\beta_q}_{r_1\cdots r_p s_1\cdots s_q}a_{\alpha_1\cdots\alpha_p}b_{\beta_1\cdots\beta_q}; \tag{5}$$

$p+q$ must be $\leqq n$ in order that the alternating tensor thus arrived at may not vanish identically.

Similarly we derive from two contravariant tensors $a^{r_1\cdots r_p}$ and $b^{s_1\cdots s_q}$ of ranks p and q respectively an alternating contravariant tensor of rank $p+q$

$$c^{r_1\cdots r_ps_1\cdots s_q} = \delta^{r_1\cdots r_ps_1\cdots s_q}_{\alpha_1\cdots\alpha_p\beta_1\cdots\beta_q}a^{\alpha_1\cdots\alpha_p}b^{\beta_1\cdots\beta_q}. \tag{6}$$

If the original tensors are alternating considerable simplification results. It is convenient to remove a numerical factor $p!\,q!$ and we may define the derived tensor $c_{r_1\cdots r_ps_1\cdots s_q}$ by the equations

$$\begin{aligned} c_{r_1\cdots r_ps_1\cdots s_q} &= \frac{1}{p!}\frac{1}{q!}\delta^{\alpha_1\cdots\alpha_p\beta_1\cdots\beta_q}_{r_1\cdots r_ps_1\cdots s_q}a_{\alpha_1\cdots\alpha_p}b_{\beta_1\cdots\beta_q}, \\ &= \sum_{l,\,m}\delta^{l_1\cdots l_pm_1\cdots m_q}_{r_1\cdots r_ps_1\cdots s_q}a_{l_1\cdots l_p}b_{m_1\cdots m_q}. \end{aligned} \tag{7}$$

In the last expression $(l_1, l_2, \cdots, l_p)$ is any group of p out of the $p+q$ numbers $(r_1, \cdots, r_p, s_1, \cdots, s_q)$ and $(m_1, \cdots, m_q)$

is the remaining group of q numbers. No group $(l_1, \cdots, l_p)$ is to be repeated in the summation. The tensor derived in this way may be called the outer product of the two alternating covariant tensors. Similarly the outer product of two alternating contravariant tensors is given by

$$(8) \qquad c^{r_1 \cdots r_p s_1 \cdots s_q} = \frac{1}{p!}\frac{1}{q!}\delta^{r_1 \cdots r_p s_1 \cdots s_q}_{\alpha_1 \ldots \alpha_p \beta_1 \ldots \beta_q} a^{\alpha_1 \cdots \alpha_p} b^{\beta_1 \cdots \beta_p}$$

$$= \sum_{l,\, m} \delta^{r_1 \cdots r_p s_1 \cdots s_q}_{l_1 \cdots l_p m_1 \cdots m_q} a^{l_1 \cdots l_p} b^{m_1 \cdots m_q}.$$

The alternating tensor derived as in (5) from two non-alternating tensors is not essentially different from the outer product of two alternating tensors. For we may write it, on using the result*

$$\delta^{l_1 \cdots l_p m_1 \cdots m_q}_{r_1 \cdots r_p s_1 \cdots s_q} = \frac{1}{p!}\frac{1}{q!}\delta^{l_1 \cdots l_p}_{\lambda_1 \ldots \lambda_p} \delta^{m_1 \cdots m_q}_{\mu_1 \ldots \mu_q} \delta^{\lambda_1 \cdots \lambda_p}_{r_1 \cdots r_p} \delta^{\mu_1 \cdots \mu_q}_{s_1 \cdots s_q},$$

in the form

$$\frac{1}{p!}\frac{1}{q!}\delta^{\lambda_1 \ldots \lambda_p \mu_1 \ldots \mu_q}_{r_1 \cdots r_p s_1 \cdots s_q}\left(\delta^{\alpha_1 \cdots \alpha_p}_{\lambda_1 \ldots \lambda_q} a_{\alpha_1 \ldots \alpha_p}\right)\left(\delta^{\beta_1 \cdots \beta_q}_{\mu_1 \ldots \mu_q} b_{\beta_1 \ldots \beta_q}\right)$$

showing that $c_{r_1 \ldots r_p s_1 \cdots s_q}$, as defined in (5), is really the outer product of the two alternating tensors

$$f_{t_1 \cdots t_p} = \delta^{\alpha_1 \cdots \alpha_p}_{t_1 \ldots t_p} a_{\alpha_1 \ldots \alpha_p}$$

and

$$g_{u_1 \cdots u_q} = \delta^{\beta_1 \cdots \beta_q}_{u_1 \cdots u_q} b_{\beta_1 \ldots \beta_q}.$$

It is well known that alternating covariant tensors may be used as the coefficients of invariant integral forms. In fact, we may, from p arbitrary contravariant tensors $a^r, b^s, \cdots, g^t$, of rank one, form the alternating contravariant tensor of rank p

$$e^{r_1 \cdots r_p} = \delta^{r_1 \cdots r_p}_{\alpha_1 \cdots \alpha_p} a^{\alpha_1} b^{\alpha_2} \cdots g^{\alpha_p}.$$

When $(a^r, b^s, \cdots, g^t)$ are differential vectors tangent to the parametric lines of a spread of p dimensions so that

* See equation $(2 \cdot 4^{\text{bis}})$, paper (A).

$$a^r = \frac{\partial x^r}{\partial u_1} du_1,$$

etc., the scalar product $a_{\varrho_1 \cdots \varrho_p} e^{\varrho_1 \cdots \varrho_p}$ is called *an integral form of order p* (in this case $e^{r_1 \cdots r_p}$ is usually denoted by $d(x^{r_1} x^{r_2} \cdots x^{r_p})$). Its integral over the spread is called *the integral of the alternating covariant tensor $a_{r_1 \cdots r_p}$ over the spread of p dimensions.* In this connection the result (7), which enables us to derive from two integral forms of orders p and q respectively an integral form of order $p+q$, is known as the *law of the outer product of two integral forms.**

3. *Connection between the Kronecker Tensor and Reciprocation with respect to a Quadratic Differential Form.* If we have any symmetric covariant tensor of the second rank a_{rs} we may denote by a the value of the determinant which has a_{rs} as the element in its rth row and sth column. It follows at once that the product $\sqrt{a}\ d(x^1 x^2 \cdots x^n)$ is invariant, and so we may introduce an alternating covariant tensor $\varepsilon_{r_1 \cdots r_n}$ which is defined by the statement that its components have the value $\pm \sqrt{a}$ according as the arrangement $(r_1, r_2, \cdots, r_n)$ of the n numbers $(1, 2, \cdots, n)$ is of the even or odd class. For the integral form

$$\varepsilon_{\alpha_1 \cdots \alpha_n} d(x^{\alpha_1} x^{\alpha_2} \cdots x^{\alpha_n})$$

is invariant, its value being $n! \sqrt{a}\, d(x^1 x^2 \cdots x^n)$. From this alternating covariant tensor we derive an alternating contravariant tensor of rank n as follows. Solving the system of n^2 equations

$$a_{r\alpha} a^{\alpha s} = \delta_r^s$$

for the n^2 unknowns a^{rs}, we obtain a contravariant tensor of rank two which is said to be the reciprocal of a_{rs} with respect to the quadratic differential form $a_{\alpha\beta} dx^\alpha dx^\beta$. The tensor $\varepsilon^{r_1 \cdots r_n} = a^{r_1 \alpha_1} a^{r_2 \alpha_2} \cdots a^{r_n \alpha_n} \varepsilon_{\alpha_1 \cdots \alpha_n}$ is said to be the

* Reference may be made to E. Cartan, *Leçons sur les Invariants Intégraux*, Paris, 1922. E. Cartan, ANNALES DE L'ECOLE NORMALE, 1899. H. Bateman, *Differential Equations*, Chapter 7. London, 1918. H. Bateman, PROCEEDINGS OF THE LONDON SOCIETY, (2), vol. 8 (1910).

reciprocal of $\varepsilon_{r_1 \cdots r_n}$ with respect to the quadratic differential form. It is alternating and has the value $\pm 1/\sqrt{a}$ according as the arrangement $(r_1, \cdots, r_n)$ of the n numbers $(1, 2, \cdots, n)$ is of the even or odd class. Then the simple product $\varepsilon^{r_1 \cdots r_n}\varepsilon_{s_1 \cdots s_n}$ is the generalised Kronecker tensor

$$\delta^{r_1 \cdots r_n}_{s_1 \cdots s_n}.$$

It may be remarked that the operation of finding tensors reciprocal to any quadratic differential form is a possible one so long as the differential form is non-singular ($a \neq 0$). When the quadratic differential form is the metrical ground form $(ds)^2 = g_{\alpha\beta}dx^\alpha dx^\beta$ of a Riemann space we may say that the reciprocal covariant and contravariant tensors are but different representations of the same physical idea which is the tensor. This is by analogy with the case of rectangular cartesian coordinates in euclidean space where the distinction between covariance and contravariance breaks down and the components of two tensors which are reciprocal with respect to the metric ground form $(ds)^2 = (dx^1)^2 + (dx^2)^2 + \cdots + (dx^n)^2$ coincide.

We have endeavored to show in the preceding paragraphs the utility of the alternating tensor $\delta^{r_1 \cdots r_m}_{s_1 \cdots s_m}$ in proving theorems of tensor algebra. It will be readily recognised that there is an intimate connection here with Grassmann's *Ausdehnungslehre*, and we believe, in fact, that a systematic exposition of this theory with the aid of the generalized Kronecker symbol would help to make it more widely understood. The use of the symbol in connection with the discussion of the orientation of cells in analysis situs* is also recommended.

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* O. Veblen. *Cambridge Colloquium Lectures*, New York, 1922.