FUNCTIONS WITH AN ESSENTIAL SINGULARITY*

BY PHILIP FRANKLIN

1. *Introduction.* In this note we prove certain properties of functions possessing essential singularities. The results grew out of an attempt to prove that the equation

$$\sin x = x$$

has an infinite number of complex roots. This particular fact can be deduced very simply from Picard's theorem (see Theorem VI below), but it suggests other inferences which are much less immediate. The character of the theorems, which are given explicitly below, may be indicated by observing that they prove the existence of an infinite number of roots of the following typical equations:

$$x^3 e^{x^2} + 3x = 0,$$

$$\cos x - \frac{2x+1}{x^2-4} = 0,$$

$$\sec x + 5x = 0.$$

2. *A Corollary to Picard's Theorem.* One possible formulation of Picard's theorem is the following:†

If the function $f(x)$ has an essential singularity at the point P, and in some deleted neighborhood of P is analytic except for a finite number of poles and has only a finite number of zeros, then the equation

$$f(x) - a = 0$$

has an infinite number of roots in the neighborhood in question for all values of a different from zero. The proposed corollary follows.

THEOREM I. *If the function $f(x)$ has an essential singularity at the point P, and in some deleted neighborhood of P is*

---

* Presented to the Society, May 3, 1924.

† Cf., e. g., Osgood, *Funktionentheorie*, 2d ed., vol. 1, p. 709.

analytic except for a finite number of poles and has only a finite number of zeros, while the function $A(x)$, not identically zero, is analytic except for poles in the complete neighborhood of P, then the equation

$$f(x)-A(x) = 0$$

has an infinite number of roots in this neighborhood.

To prove this, consider the function $f(x)/A(x)$. This function is analytic except for poles in the deleted neighborhood of P, and has an essential singularity at P. Further, it has at most a finite number of zeros and poles in the neighborhood in question, since both $f(x)$ and $A(x)$ have at most a finite number of zeros and poles there. Thus, by the theorem of Picard just stated, the equation

$$f(x)/A(x)-1 = 0$$

must have an infinite number of roots. Moreover, since $A(x)$ has only a finite number of poles, the equation

$$A(x)[f(x)/A(x)-1] \equiv f(x)-A(x) = 0$$

must in consequence have an infinite number of roots, which proves our contention.

3. *Applications to Entire Functions.* We may obtain a simplified statement of our theorem by specializing it somewhat. Let us take the point P at infinity, the deleted neighborhood as the proper plane, and require the function to have no poles. The function is then an entire function, and the theorem takes the following form.

THEOREM II. *If $E(x)$ is an entire function (not a polynomial) with only a finite number of zeros, and $R(x)$ is a rational function (not identically zero), the equation*

$$E(x)-R(x) = 0$$

always has an infinite number of roots.

If we further restrict $R(x)$ to be a polynomial, and notice that changing a finite number of the coefficients in the Maclaurin's expansion of a function is equivalent to adding a suitably chosen polynomial to it, we obtain the following remarkable theorem.

THEOREM III. *If an entire function has at most a finite number of zeros, any function formed from it by changing a finite number of the coefficients in its Maclaurin's expansion has an infinite number of zeros.*

A statement equivalent to Theorem III is the following one.

THEOREM IIIa. *If an entire function is known to have only a finite number of zeros, and the coefficients of its Maclaurin's expansion from a certain point on are known, the preceding ones are uniquely determined.*

It is interesting to observe that the assumption that there actually exists an expansion with the desired properties is necessary for the truth of the theorem. That is, we can not start with any infinite expansion, and determine the earlier coefficients so as to give a function with only a finite number of zeros, as one might suppose. For example, if the terms following the mth are

$$K(x) = \sum_m^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!},$$

no choice of the earlier terms will give a function which does not have an infinite number of zeros. This follows from Theorem V below, since $K(x)$ is the remainder in the expansion of sin x.

4. *The Generalized Corollary.* There is an extension of our corollary, analogous to the extension of Picard's theorem to the case in which the function has an infinite number of poles and fails to take on two values more than a finite number of times. The extension is as follows.

THEOREM IV. *If the function $f(x)$ has an essential singularity at the point P, which is a cluster point of poles, otherwise being analytic in some neighborhood of P, and takes on two distinct values a and b at most a finite number of times in this neighborhood, while the function $A(x)$ is analytic* *except for poles in the complete neighborhood of P, and is not identically equal to a or b, then the equation*

$$f(x) - A(x) = 0$$

has an infinite number of roots in the neighborhood in question.

To prove this, we first observe that the function

$$\frac{f(x)-a}{f(x)-b} = 1 + \frac{b-a}{f(x)-b}$$

has an essential singularity at P (since a and b are distinct), and that it has at most a finite number of zeros and poles by our hypothesis as to $f(x)$. Likewise the function

$$\frac{A(x)-a}{A(x)-b}$$

is analytic except for poles in the complete neighborhood of P, and is not identically zero. Thus the equation

$$\frac{f(x)-a}{f(x)-b} - \frac{A(x)-a}{A(x)-b} = 0$$

satisfied the hypothesis of Theorem I, and hence has an infinite number of roots.

Next consider

$$\frac{[f(x)-b]\,[A(x)-b]}{a-b}\left[\frac{f(x)-a}{f(x)-b} - \frac{A(x)-a}{A(x)-b}\right] = 0.$$

Any value of x which satisfies the previous equation will satisfy this one, unless it is a pole of one of the outside factors. But such a value would have to be a pole of $f(x)$ or $A(x)$, and from the form of the previous equation, we see that it would have to be a pole of both, if of either. Thus all the roots lost are included in the poles of $A(x)$, and since this function has at most a finite number of poles, the equation last written has an infinite number of roots. Since this equation reduces to

$$f(x) - A(x) = 0,$$

the theorem is proved.

Both Theorems I and IV may be slightly strengthened by replacing the function $A(x)$ by two functions. The revised statement of Theorem IV is as follows:

THEOREM IVa. *If the function $f(x)$ has an essential singularity at the point P, which is a cluster point of poles, otherwise being analytic in some neighborhood of P, and takes on two distinct values a and b at most a finite number of times in this neighborhood, while the functions $A(x)$ and $B(x)$ are analytic except for poles in the complete neighborhood of P, and $A(x)/B(x)$ is not identically equal to a or b, then the equation*

$$B(x)f(x)-A(x) = 0$$

has an infinite number of roots.

This follows at once from Theorem IV, since that theorem may be applied to the equation $f(x)-A(x)/B(x) = 0$. The analogous strengthened form of Theorem I is somewhat similar to a theorem given by Borel for entire functions of finite genus, where our functions $A(x)$ and $B(x)$ are required to be polynomials.*

5. *Applications to Periodic Functions.* A periodic function which is not a constant must have an essential singularity at infinity. Hence the preceding theorems may be used to derive theorems about periodic functions. In fact, the periodicity enables us to weaken the hypothesis. We have, for example, the following theorem.

THEOREM V. *If $f(x)$ is periodic, is not a constant, and has no poles, while $A(x)$ is analytic except for poles in some complete neighborhood of infinity, and is not a constant, then the equation $f(x)-A(x) = 0$ has an infinite number of roots in the neighborhood in question.*

If the theorem were false, the equation

$$f(x)-A(x) = 0$$

would have only a finite number of roots, and its left member would satisfy all the conditions required of

* *Leçons sur les Fonctions Entières,* 1921, p. 90.

the function $f(x)$ of Theorem I. Likewise the function $A(x+p)-A(x)$, where p is the period of $f(x)$, would satisfy all the conditions required of the function $A(x)$ of that theorem. It is not identically zero since $A(x)$, being analytic or having a pole at infinity, and not being a constant, is not periodic. Thus by Theorem I, the equation

$$[f(x)-A(x)]-[A(x+p)-A(x)] = 0$$

would have an infinite number of roots. But, since

$$f(x) = f(x+p),$$

this equation is equivalent to

$$f(x+p)-A(x+p) = 0,$$

which accordingly has an infinite number of roots. Since this would necessitate the existence of an infinite number of roots for the equation

$$f(x)-A(x) = 0,$$

it contradicts our assumption, and the theorem is proved.

For one special choice of the function $A(x)$, we may weaken the restriction on $f(x)$ still further, so as to obtain the following theorem.

THEOREM VI. *If $f(x)$ is periodic and not a constant, the equation*

$$f(x)-ax = 0$$

always has an infinite number of roots.

For, if the equation $f(x)-ax = 0$ had only a finite number of roots, so would

$$f(x+np)-a(x+np) = 0,$$

p being the period of $f(x)$, and n being any integer. Since this is equivalent to $f(x)-ax = anp$, we would have a function with an essential singularity which failed to take on all the values anp more than a finite number of times, which contradicts Picard's theorem.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY