RICCI'S COEFFICIENTS OF ROTATION*

BY HARRY LEVY

In a Riemann space whose linear element is given by the positive definite quadratic form

$$(1)\qquad ds^2 = g_{ij}dx^i dx^j,$$

let us take n congruences of curves, defined by the equations

$$(2)\qquad \frac{dx^1}{\lambda_{h}|^1} = \frac{dx^2}{\lambda_{h}|^2} = \cdots = \frac{dx^n}{\lambda_{h}|^n} \quad (h = 1, 2, \cdots, n).$$

Let us denote, as usual, the covariant components of congruence $\lambda_{h|}$ by $\lambda_{h|r}$, so that

$$(3)\qquad \lambda_{h|r} = g_{rt}\lambda_{h}|^t.$$

We assume, further, that these n congruences are mutually orthogonal. We may then write

$$(4)\qquad g_{rt}\lambda_{h}|^r \lambda_{k}|^t = \delta_{hk},$$

where δ_{hk} is Kronecker's delta, that is,

$$(5)\qquad \delta_{hk} = \begin{cases} 0, & h \neq k, \\ 1, & h \neq k. \end{cases}$$

Ricci† has found a set of invariants which he calls *coefficients of rotation* which are very important in the physical and geometric applications. They are defined by the equations

$$(6)\qquad \gamma_{hij} = \lambda_{h|r,t}\lambda_{i}|^r \lambda_{j}|^t,$$

where $\lambda_{h|r,t}$ is the covariant derivative of $\lambda_{h|r}$,

$$(7)\qquad \lambda_{h|r,t} = \frac{\partial \lambda_{h|r}}{\partial x^t} - \lambda_{h|p}\Gamma_{rt}{}^{p},$$

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† *Dei sistemi di congruenze ortogonali in una varietà qualunque*, MEMORIE DELLA R. ACCAD. DEI LINCEI, CLASSE DEI SCIENZE (5), vol. 2 (1896). See also *Méthodes de calcul différentiel absolu*, by Ricci and Levi-Civita, MATHEMATISCHE ANNALEN vol. 54 (1901), p. 147 ff.

where Γ_{rt}^{p} is the Christoffel symbol of the second kind of the fundamental form (1).

Lipka* has obtained the geometric interpretation of these functions γ_{hij} and the conditions that a congruence of curves be parallel in the sense of Levi-Civita.† In this paper I shall obtain Lipka's results in another way, and I shall also prove a new theorem.

Take an arbitrary curve, C, of congruence $\lambda_{j|}$ and fix on it some point P. Let $\mu_i|^r$ for $r = 1, 2, \cdots, n$ be the contravariant components of the vector which is parallel to itself along C and which at P coincides with the tangent vector to the curve of congruence $\lambda_{i|}$. Then, by the definition of parallelism, we have

$$\mu_i|^r{}_{,t}\lambda_j|^t = 0 \tag{8}$$

along C, where $\mu_i|^r{}_{,t}$ is the covariant derivative of $\mu_i|^r$. Let ω_{hi} be the angle, at the points of C, between the vectors $\lambda_{h|}$ and $\mu_{i|}$. Then

$$\cos\omega_{hi} = \lambda_{h|r}\mu_i|^r. \tag{9}$$

Differentiating covariantly with respect to x^t, we find

$$\frac{\partial \cos\omega_{hi}}{\partial x^t} = \lambda_{h|r,t}\mu_i|^r + \lambda_{h|r}\mu_i|^r{}_{,t}.$$

Multiplying by $\lambda_j|^t$, summing on t, and taking into account equations (8), we obtain

$$\frac{\partial \cos\omega_{hi}}{\partial s_j} = \lambda_{h|r,t}\mu_i|^r\lambda_j|^t, \tag{10}$$

where s_j is the arc of C. From (6) and from the definition of $\mu_{i|}$ it follows that at P we have

$$\left.\frac{\partial \cos\omega_{hi}}{\partial s_j}\right|_P = \gamma_{hij}.$$

* *On Ricci's coefficients of rotation*, JOURNAL OF MATHEMATICS AND PHYSICS, MASSACHUSETTS INSTITUTE OF TECHNOLOGY, vol. 3, 1924.

† *Nozione di parallelismo in una varieta qualunque*, RENDICONTI DI PALERMO, vol. 42 (1917). Also Bianchi's paper in RENDICONTI DI NAPOLI, (3a), vol. 27 (1922).

Hence γ_{hij} *is the rate of change along a curve of congruence* $\lambda_{j|}$ *of the cosine of the angle between the curves of* $\lambda_{h|}$ *and the parallels, with respect to* $\lambda_{j|}$, *to the curves of* $\lambda_{i|}$.

Since $\gamma_{hij} = -\gamma_{ihj}$,* the angle between the curves of $\lambda_{h|}$ and the parallels to $\lambda_{i|}$ changes at the same rate as the angle between the curves of $\lambda_{i|}$ and the parallels to $\lambda_{h|}$. In a euclidean space $\partial(\cos\omega_{hi})/\partial s_j$ is exactly what we would call the rotation of the congruence $\lambda_{i|}$ about $\lambda_{h|}$.

Suppose congruence $\lambda_{i|}$ forms a system of parallels with respect to congruence $\lambda_{j|}$, that is, those curves of $\lambda_{i|}$ which intersect one and the same curve of $\lambda_{j|}$ are parallel with respect to that curve, and this holds along all curves of $\lambda_{j|}$; then, $\mu_{i|}$ coincides with $\lambda_{i|}$, and

$$\cos\omega_{hi} = \delta_{hi}, \qquad (h = 1, 2, \cdots, n).$$

Hence

$$\gamma_{hij} = 0, \qquad (h = 1, 2, \cdots, n), \tag{11}$$

is the necessary condition that the congruence $\lambda_{i|}$ forms a system of parallels with respect to $\lambda_{j|}$. This is also sufficient. For from equations (6) we have

$$\lambda_{h|r,t} = \sum_{i,j=1}^{n} \gamma_{hij} \lambda_{i|r} \lambda_{j|t}.$$

By virtue of (3) and (4), (10) become

$$\frac{\partial \cos\omega_{ih}}{\partial s_j} = \sum_{l=1}^{n} \gamma_{ilj} \lambda_{l|r} \mu_h|^r = -\sum_{l=1}^{n} \gamma_{lij} \lambda_{l|r} \mu_h|^r.$$

If (11) are satisfied, we have

$$\frac{\partial \cos\omega_{ih}}{\partial s_j} = 0, \qquad (h = 1, 2, \cdots, n).$$

Hence ω_{ii} is constant, and therefore $\lambda_{i|}$ is parallel along $\lambda_{j|}$.

The author has shown† that the necessary and sufficient conditions that a congruence $\lambda_{k|}$ have a family of m-dimensional hypersurfaces as orthogonal trajectories is that

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* Ricci and Levi-Civita, loc. cit., p. 148.

† *Normal congruences of curves,* this BULLETIN, vol. 31, p. 39.

$$\gamma_{hij} = \gamma_{jih} \tag{12}$$

where i takes on a definite set of $n-m$ values of the integers $1, 2, \cdots, n$ including the value k, and h and j take on those m values that i cannot assume. If each of $n-m$ congruences forms a system of parallels with respect to every one of the remaining m congruences equations (11) are satisfied for $h = 1, 2, 3, \cdots, n$; $i = i_1, i_2, \cdots, i_{n-m}$; $j = j_1, \cdots, j_m$; $i \neq j$, (12) is surely satisfied, and we have the following theorem.

THEOREM. *If each of $n-m$ congruences forms a system of parallels with respect to every one of the remaining m congruences, then the former have a family of m-dimensional hypersurfaces as orthogonal trajectories.* When $m = n-1$ this reduces to one of Lipka's theorems.

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SUR LES VALEURS ASYMPTOTIQUES DES COEFFICIENTS DE COTES

PAR J. OUSPENSKY

1. Parmi les formules de quadratures pour le calcul approché des intégrales définies la plus simple est, sans contredit, celle de Cotes, qui correspond à la division de l'intervalle d'intégration en parties égales. Supposons l'intervalle d'intégration (0, 1) subdivisé en n parties égales; alors on peut déterminer $n+1$ constantes $A_0, A_1, A_2, \cdots, A_n$, nommées "coefficients de Cotes", de manière que la formule

$$\int_0^1 f(x)\,dx = A_0 f(0) + A_1 f\left(\frac{1}{n}\right) + A_2 f\left(\frac{2}{n}\right) + \cdots + A_n f(1)$$

soit exacte pour toute fonction $f(x)$ se réduisant à un polynôme d'un degré n'excédant pas $n-1$. Dans d'autres cas cette "formule de Cotes" n'est qu'approchée. Comme le degré d'approximation fourni par elle depend des valeurs numériques des coefficients $A_0, A_1, A_2, \cdots, A_n$, la