STUDY ON VECTORS AND INVARIANTS

Einleitung in die Theorie der Invarianten linearer Transformationen auf Grund der Vektorenrechnung. Erster Teil. By E. Study. Braunschweig, Vieweg und Sohn, 1923. 268 pp.

Like other writings of Professor Study the present work is interesting from several different angles. It is a development of the theory of invariants of ternary forms, based upon what he calls vectors; it is an exposition of his thesis that mathematics is the study* of "natural (positive, integral) numbers and everything which can be based upon them, but nothing else"; it is a running commentary and criticism of many things and many mathematicians. One needs occasionally a background of other papers of his dealing with questions of a semi-philosophical character. He particularly takes to task the investigators who have developed systems of vector analysis because they have not contented themselves with utilising the theory of invariants, as developed by Clebsch, Aronhold, etc. He says (p. 3) "Whether one develops vector analysis for the sake of its applications or as a self-contained discipline, there must always come first, as stated, a consideration of certain topics of algebra, namely invariants of certain groups of linear transformations: all expressions considered are invariants of orthogonal transformations, or invariants of groups closely related to the group of orthogonal transformations. But for more than fifty years we have had a highly developed theory of the group of all linear transformations, and for more than twenty-five years at least the fundamentals of a theory of invariants of the other groups just referred to (Ges. Wiss. Leipzig, Math.-Phys. 1897, p. 443 ff.). But not a glimmer of light seems to have fallen from these investigations upon the highly beloved "Vector Analysis" of today. Indeed old problems have been handled as if never treated before, and we are thus lagging far behind what has long been a guaranteed possession of the science. So far as I know, the question is never raised in such writings, as to what are all possible algebraic, and in particular, rational, invariants of these groups; and yet there can be no doubt that this is the problem, which from the very nature of things, lies at the heart of the matter." And again "As for the majority of authors it is not evident that they lived in a generation when the theory of groups was in full bloom. We even see the algorithm of H. Grassmann, which was a mark of progress in his own time, and which stands as a monument to his originality, yet which has long ago been absorbed into the more profound theory of invariants, hailed as the acme of the attainable, or even as a panacea, which certainly it is not; while on the

* *Mathematik and Physik*, p. 6.

other hand algebraists like Aronhold and Clebsch, who, differing from their predecessors, have set out from precisely stated problems, and have gone much farther, do not seem to have existed for these same authors. In short we are behind the times, and nothing less."

The Introduction, of twelve full pages, is labeled "Problems and methods", and is an insistence upon the importance of problems and the relative unimportance of methods. "Those who construct vector analysis and related algorithms whose essence is method (extensive algebra, calculus of matrices, quaternions, etc.) usually consider their subject-matter as a world in itself. Doubtless there is much justification for this. Such a claim and the related desire for purity of method contain a wholesome pressure towards thoroughness". There are esthetic considerations but there is also the usefulness and ease of manipulation of an algorithm to be taken into account. A happy notation is a great advantage in the progress of a new branch of mathematics, but for that very reason one must hesitate to introduce novelties merely for their own sake, or from a mere desire not to "violate the spirit of the order". The reader will not usually learn a new notation thoroughly, and will in the end give up trying to translate it into what he is familiar with. This also has to be said with regard to the invention of new terms, as, for instance, calling ternary bilinear forms, or the system of coefficients that belong to such a form, by the new terms: tensor of the second kind, affinor, dyadic, tensortriple, complete dyad, asymmetric tensor, diatensor, vector homography, and special terms related, as deviator, antitensor, axiator, idemfactor, versor, perversor, etc. And "what has long been sought in vector calculus—a system of symbols most convenient for the subject,—has been in existence a long time, in a form much more convenient than anything devised since by the various partisans. I shall make use here with minor alterations (whose motive should not be missed) of the notation based upon the older theory of invariants of linear transformations, which is uniquely determined in all essentials by the subject itself. A special notation for vectors, which only too easily may become an obstacle in the free road, seems to me superfluous if not harmful. The 'world for itself' will nevertheless come into its full rights." Professor Study thus comes to the conclusion that the problems of vector calculus are merely invariant problems, and the methods of invariants are the most satisfactory for all purposes of vectors. Thus the raison d'être of vector algorithms has disappeared!

Professor Study also has no uncertain opinion with regard to many ideas of many mathematicians. On page 3, Grassmann is said to have been "absorbed". On page 4 an unnamed writer* is criticised for apologiz-

* See Weyl, MATHEMATISCHE ZEITSCHRIFT, vol. 20 (1924), p. 131.

ing for the deluge of formulas he introduces. On page 5 another unnamed writer is criticised for his remark relative to Hilbert's theorem on the finiteness of systems of forms: "How fine that we need not bother any more with invariants." One must feel that perhaps the point was missed. On page 8, Boltzmann is mentioned as saying: "One should leave elegance to cobblers and tailors". On page 11, Beck is criticised for his *Coordinate Geometry*, though he certainly is in sympathy with Study's ideas. On page 19 and page 21, Lie is mentioned as illogical for his term "continuous group", since Study desires to use the term in another sense. On page 52, Coolidge gets favorable mention, as do Veblen and Young on page 59. On page 101, Grassmann's Lückenprodukte are condemned. On the same page the mathematical ability of Einstein is questioned, and Weyl's notations are condemned. On page 118, the right of authors to invent terms is questioned. On page 127, Frobenius comes in for a share of criticism, along with unnamed mathematicians who use the English language, and spend time on hypernumbers. The criticism of Frobenius' standpoint extends over to page 129. On page 238, authors of analytic geometries are criticised. On page 257, Rabinovitch and the Italian school get their deserts.

Professor Study's definitions remove geometry from the study of space, as usually conceived, completely. For him a point is merely a triplex of numbers, or in general a multiplex of N numbers, and a vector is precisely the same thing. The term multiplex, it may be explained, means a set of numbers in a specified order. It is exactly the idea of Hamilton over again, who first dealt with couples, then with "sets" of numbers. Of course Study would and does repudiate the Hamilton relation of number to "time", but as a matter of fact Hamilton's "time" is nothing more than the linear continuum. He does not point out, however, that it was Hamilton who first made a study of the couple, multiplex set, etc., as an entity. He did not call it a vector, defining (for the first time) this to mean a geometric segment with a definite direction. One who insists on the correct use of established terminology should have retained the very useful terms of Hamilton. The reviewer agrees perfectly with Professor Study that good existing terminologies should be preserved, and new ones avoided; therefore he desires Professor Study to follow the same caution. The same remark applies to changing the terminology of Lie. And it is to be noted further that one does not escape axioms and postulates by merely referring everything to the system of natural numbers. If that were possible he could go still further and start with Whitehead and Russell's *Principia.* We might grant that it is possible to set up a useful isomorphism between points or vectors in space and triplexes, but it is easily evident that the isomorphism will work both ways, and that arithmetic can be based upon space just as much as space can be based upon arithmetic. A good deal of so-called simplification and "going to the foundations" consists in substituting one abstraction for another.

But we also have a conviction that the theory of vectors is not the same by any means as the theory of invariants. Professor Study seems to have forgotten the origin of quaternions, and from it that of the various systems of vectors. If one reads Hamilton carefully he will find that Hamilton was concerned with the extensions of number along the lines of algebra, starting with negative numbers and the so-called imaginary. In other words he was studying the theory of hypernumbers and made considerable progress along this line. These are not invariants, any more than other numbers are invariants, and their development antedates that of invariants by several years. The idea of linear transformations was superimposed upon them and is not the outcome of their study. It is only in a very superficial sense that one could consider that the idea of negative, or of $\sqrt{-1}$, was an outcome of natural number. These notions may be called with some justification an outcome of operations upon number, but operations upon number and number itself are very different entities. Further to represent $\sqrt{-1}$ as a linear substitution, as Peano does, is again only an isomorphism, and should not be confused with an identity. Keeping this in mind we will see that a system of vectors is a system of algebraic numbers in an extended sense, and all formulas are algebraic formulas, when algebra is thus understood. They do not primarily deal with multiplezes at all.* A quaternion is an algebraic number of the type of the ordinary complex number, and so long as we deal only with its powers and their combinations with positive and negative numbers, a quaternion differs in no sense from a complex number. Further it is an individual entity and not a set of four numbers. This Hamilton very well understood though he used sets of four positive or negative numbers in the various representations of quaternions. When this fact is clearly understood all the nonsense about geometric numbers and the like vanishes. The expressions in the formulas of such hypernumbers are not there because of certain invariancies, but because of the properties of these algebraic numbers. That they turn out to be invariants for orthogonal or other transformations of the so-called coefficients is due to such facts as that in any quaternion formula as $S\alpha\beta$, or Sqr, we have such relations as

$$S\cdot t\alpha t^{-1} t\beta t^{-1} = S\cdot \alpha\beta, \qquad S\cdot tqt^{-1}trt^{-1} = Sqr.$$

Therefore when Professor Study laments the fact that investigators have not tried to list the sets of "fundamental invariants" he might as well be concerned over the fact that not much study has been made

* The „multiplex" character means merely that from *one* equation between vectors arise an *infinity* of equations between ordinary numbers, all dependent on any N of the set which are linearly independent.

of the arithmetics that are developed in such hypernumber systems, though the latter line of development is now started by Dickson. Hence the charge that the theory of invariants has not existed for such investigators is not well-founded. And further in the application of such hypernumbers to geometric problems no concern need be felt over invariants, for where the expressions do not depend upon coordinate planes, they are ipso facto all invariant expressions. The reviewer has pointed this out elsewhere* with reference to the Einstein relativity theory, showing that when such theory is stated in terms of an N-dimensional vector calculus, there are no coordinates involved, hence a change of coordinate references has nothing to do with the problem. The thesis that all physical phenomena should be so represented that the same expressions will hold whatever the system of coordinates is actually realised by removing all systems of coordinates. Indeed the difficulties dragged in by coordinates have no business there at all. And to assert, as Professor Study does, that there is no geometry of a sound logical character without coordinates, is to ignore the whole of Euclid, or any other system of synthetic geometry. And to assert further that this is mixing up the phenomenal world, and questions of epistemology, with geometry, is to show that one has not yet clearly seen what is going on in the study of synthetic geometry. Space forbids entering into a full explanation, but we may point out that the mind studies many types of ideal or non-material "constructions", and among these we find the ones that enter geometry alongside of the "arithmetic", and one has the same kind of reality or validity as the other.

The reviewer is quite well aware of all the notions that have clung to that of vector calculus, and has always maintained that those systems that were mere shorthand or symbolisms for coordinate expressions were not worthy the name of vector calculus. Nothing can be accomplished by them which cannot also be done equally well without them, and in so far Professor Study is right in insisting that if one utilises the algebra of invariants he can get along just as well. There is of course some small gain in the use of the vectors but it is a matter of taste whether one prefers to write either of the two sets of expressions (the first is Study's, the second is quaternions)

$$(AP) \text{ or } m_1(\varphi),\ (AP')(A'P) \text{ or } m_1(\varphi^2),\ (AP')(A'P'')(A''P) \text{ or } m_1(\varphi^3),$$
$$\tfrac{1}{6}(AA'A'')(PP'P'') \text{ or } m_3(\varphi),\ (XU) \text{ or } S\rho\tau,\ (XA)(PU) \text{ or } S\rho\varphi\tau,$$
$$(XA)(PA')(P'U) \text{ or } S\rho\varphi^2\tau,\ \tfrac{1}{2}(XPP')(AA'U) \text{ or } S\rho\psi(\varphi)\tau,\ (SX)^3$$
$$\text{or } S\rho\varphi\rho\varphi^2\rho,\ (\Sigma U)^3 \text{ or } S\tau\breve{\varphi}\tau\breve{\varphi}^2\tau \text{ etc.}$$

One form is no more intelligible than the other, and expresses no more than the other so long as we are planting our feet on the artificial

* *General vector algebra*, TRANSACTIONS OF THIS SOCIETY, vol. 24 (1922), pp. 195–244.

ground of coordinates. But when we remember that the invariant symbols have no meaning for metrical work till the coordinate planes are given and the coordinates known, while the quaternion symbols are directly interpretable without any such artificial reference, the two sets come to have a quite different significance. There is at least as much advantage in the quaternion case as in the use of trigonometric solutions over these of rectangular coordinate geometry.

The essence of the matter is that when one is looking at every problem in geometry from the projective point of view, then projective invariants will be useful. When, however, he is looking at geometry from the metric point of view, then the direct metric analysis is far more advantageous. To use the artificial relations Study has on pages 35 et seq. is to be forgetful of the fact that mathematics is not so poverty-struck as to possess but one garment.

In order to be absolutely clear and definite, a word will not be amiss as to the significance of expressions to a quaternionist (whatever they may mean to others who merely dabble in vector analysis). The expression

$S\alpha\beta$ means the lengths of α and β multiplied by the cosine of their exterior angle;

$V\alpha\beta$ means the vector perpendicular positively to the plane α, β, whose length is the product of their lengths by the sine of the exterior angle;

$S\alpha\beta\gamma$ is the volume of the parallelepiped whose edges are α, β, γ;

$\varphi\rho$ is a linear vector function of ρ, that is, is such that we have $\varphi(x\alpha + y\beta) = x\varphi\alpha + y\varphi\beta$.

In other words $S\alpha\beta$ *does not mean* $-ax-by-cz$, $S\alpha\beta\gamma$ *does not mean* a three-rowed determinant, etc. This is what vector analysts like Burali-Forti and Marcolongo and their predecessors mean by "*autonomous expressions*", in the sense of not being related in any way to coordinate systems. Apparently some men have not yet seized this simple idea.

As far as the work before us has a constructive purpose,—that of developing a systematic text which shall be an introduction or even a manual of the theory of invariants under the linear homogeneous group or some of its subgroups, using the notion of triplex as a foundation,—we find a successful accomplishment of the purpose. The author has shown, what other writers on invariants have not, that invariants can be used, together with the symbolic notation peculiar to invariants, for the study of any geometric problems expressible by ternary forms. In a succeeding volume he expects to continue into n-ary forms. Professor Study has memoirs of long standing on these subjects, and has produced a very satisfactory book for these uses. A detailed list of contents is not necessary; the treatment is quite complete. If any book can resurrect the theory of invariants this one will.

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