THE FREQUENCY LAW OF A FUNCTION OF ONE VARIABLE*

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1. *Introduction.* If the probalility that a variable X will take on a value not greater than x is

$$\Phi(x) = \int_{-\infty}^{x} \varphi(x)\,dx,$$

then $\Phi(x)$ is the cumulative frequency law, "Verteilung", for X; whereas the frequency law is $\Phi'(x)$, which is equal to $\varphi(x)$ at a point of continuity of $\varphi(x)$. Two closely related problems will be treated in this paper:
(1) Given the frequency law $\varphi(x)$ for a variable X, to find the frequency law $\psi(y)$ for a function Y of X;
(2) Given $\varphi(x)$ and $\psi(y)$, to find Y.

Under (1), where $Y = f(X)$ is given as a continuous increasing function, no special difficulty arises.† When, however, $f(X)$ has an infinitely multiple-valued inverse $g(Y)$, the expression naturally assignable to $\psi(y)$ will not be valid without restriction.

Two real functions $\varphi(x)$ and $f(x)$ will be introduced defined for all real values of x. In case a given $\varphi(x)$ or $f(x)$ is undefined outside a finite interval, the value zero may be assigned to it outside this interval.

As the foregoing problems belong essentially to general analysis, the two theorems to be stated will avoid the language of probability.

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* Presented to the Society, December 30, 1924.

† Mayr, *Wahrscheinlichkeitsfunktionen und ihre Anwendungen*, MONATSHEFTE FÜR MATHEMATIK UND PHYSIK, vol. 30 (1920), pp. 17–43. Rietz, *Frequency distributions obtained by certain transformations of normally distributed variates,* ANNALS OF MATHEMATICS, (2), vol. 23 (1922), pp. 292–300.

2. *The Determination of the Frequency Law for a Function.*

THEOREM I. *Let*

(1) $$z = \varphi(x)$$

be a single-valued real function of x, integrable over the real continuum. Let

(2) $$y = f(x)$$

be a continuous single-valued real function, defined for all real values of x, with an inverse function,

(3) $$x = g(y),$$

in general multiple-valued. In a formal manner, set

(4) $$\psi(y) = \sum \varphi[g(y)]\ |g'(y)|,$$

where a term is to appear in the sum for each x corresponding to the given y, and where $\psi(y) = 0$ if there is no such x; and set

(5) $$\Psi(\eta) = \int_{f \leqq \eta} \varphi(x)\, dx,$$

this definite integral to be taken over those portions of the x-axis for which $f(x) \leqq \eta$, and to be zero if $f(x) > \eta$ identically. Given a real number η for which $f(x) - \eta$ has at least one zero, suppose that $\varphi[g(\eta)]$ is continuous for each corresponding x. Postulate, further, either Condition A or Condition B, below. Then we shall have

(6) $$\Psi'(\eta) = \psi(\eta).$$

CONDITION A. $f(x) - \eta$ vanishes for but a finite number of values x_i of x; and each $f'(x_i)$ exists and is not zero.

CONDITION B. (1) There is a constant M such that for small enough $\Delta\eta$

(7) $$|g'(\eta + \theta\Delta\eta)| < M, \qquad |\theta| \leqq 1.$$

(2) The lengths of intervals between consecutive zeros of $f(x) - \eta$ have a positive lower bound b.

(3) $\varphi(x)$ has but a finite number of discontinuities.

(4) When $|x|$ is sufficiently large, $\varphi(x)$ does not change sign, and $|\varphi(x)|$ does not increase with $|x|$.

PROOF. Under Condition A and also under Condition B, because of 2^0, the intersections x_i of $y=f(x)$ with $y=\eta$ are isolated. The curve—see 1^0—cuts $y=\eta$, and is not merely tangent to it. We may suppose, then, that for x_i with even subscripts, $g'(\eta)\leqq 0$, and for odd subscripts $g'(\eta)\geqq 0$. With i even, take v_i so that $x_i<v_i<x_{i+1}$,—this becomes simply $x_i<v_i$ in case x_i is the greatest abscissa of intersection. Set

$$(8)\qquad u_i=\int_{x_i}^{v_i}\varphi(x)dx;\qquad u_{i+1}=\int_{v_i}^{x_{i+1}}\varphi(x)dx.$$

Then from (5), using all values of i involved, odd and even,

$$(9)\qquad \psi(\eta)=\sum u_i.$$

But, since $\varphi(x)$ is continuous at x_i,

$$(10)\qquad u_i'=\frac{du_i}{d\eta}=\varphi(x_i)\left|\frac{dx_i}{d\eta}\right|.$$

Under Condition A, then, (6) follows from (3), (4), (9), and (10).

Now, under Condition B, we may take $\eta\Delta$ so small that, on account of 1^0, 3^0, $\varphi(x)$ is continuous when $\eta-\Delta\eta\leqq y\leqq\eta+\Delta\eta$. In (8) and (9) we may now think of η as replaced by any y in $(\eta-\Delta\eta,\ \eta+\Delta\eta)$ to form terms u_i with derivatives u_i'. Suppose that 4^0 is applicable for $x\geqq x_n-b$. Then, by (7), (10), and 2^0,

$$(11)\qquad \sum_n^\infty |u_i'|<M\sum_n^\infty|\varphi(x_i)|\leqq\frac{M}{b}\left|\int_{x_n-b}^{\infty}\varphi(x)dx\right|.$$

And, since a similar inequality can be set up for terms with $i\leqq -n$, and $\varphi(x)$ is integrable, $\sum u_i'$ converges; and also $\sum u_i$ converges. Thus (6) is also valid under Condition B, by virtue of a theorem* of function theory.

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* Dini, *Grundlagen für eine Theorie der Functionen einer veränderlichen reellen Grösse*, Leipzig, 1892, p. 154.

Porter, *On the differentiation of an infinite series term by term*, ANNALS OF MATHEMATICS, (2), vol. 3 (1901), pp. 19–20.

If (6) fails for not more than a finite number of values of η

$$(12) \qquad \Psi(\eta) = \int_{-\infty}^{\eta} \sum \varphi[g(y)] \, | \, g'(y) \, | \, dy;$$

and, indeed, less restrictive conditions may be given.

To show that the conditions of the theorem are not altogether superfluous, let us form a block diagram, and then round off slightly the corners to make it the graph of a single-valued function, — a graph very roughly representing the wave curve of damped vibration. Suppose, then, that rectangles of unit base are placed alternately above and below the X-axis, resting upon this axis, and each with a vertex upon $x^2y^2 = 1$, except near the origin. And suppose that $\varphi(x) = |x|^{-3/2}$ except near the origin. Then $\Psi'(0) = +\infty$; but $\psi(0) = 0$, since at each intersection, $g'(0) = 0$. The "curve" just described may be modified so as to oscillate about an infinity of lines parallel to the X-axis; and thus for an unlimited number of values in a finite interval $\Psi'(y) \neq \psi(y)$.

The application of the theorem to probability is obvious. The $\Psi(\eta)$ and $\Psi'(\eta)$ in (5) and (6) are respectively the cumulative frequency law and the frequency law or "frequency density" at η for the function $Y = f(X)$, when X is subject to $\varphi(x)$. E. g., if $Y = \cos X$, and $\varphi(x) = 1/2 e^{-|x|}$,

$$(13) \qquad \psi(y) = \frac{e^{-x_0} + e^{-x_1}}{1 - e^{-2\pi}} \, \frac{1}{\sqrt{1-y^2}};$$

$x_0 = \text{arc} \cos y$, $0 \leqq x_0 < \pi$, $x_1 = \text{arc} \cos y$, $\pi \leqq x_1 < 2\pi$, for $|y| < 1$; and $\psi(y) = 0$ for $|y| > 1$.

3. *The Determination of a Function connecting two Frequency Laws.*

THEOREM II. *For* $a \leqq x \leqq b$, $\alpha \leqq y \leqq \beta$, *where* a, b, α, β, *are constants, finite or infinite, let* $\varphi(x)$ *and* $\psi(y)$ *be positive except possibly for isolated values of* x *or* y; *and let these functions have finite integrals. Set*

$$(14) \qquad \Phi(\xi) = \int_a^{\xi} \varphi(x)dx; \qquad \Psi(\eta) = \int_{\alpha}^{\eta} \psi(y)dy.$$

Suppose that for some $c \leq b$,

$$\Psi(\beta) = \Phi(c). \tag{15}$$

Then there exists a single-valued continuous increasing function,

$$Y = F(X), \tag{16}$$

defined in the interval (a, c) *such that for any* ξ *in* (a, c),

$$\Psi(\eta) = \Phi(\xi); \qquad \eta = F(\xi). \tag{17}$$

PROOF. Set

$$u = \Psi(\eta). \tag{18}$$

Then u is a continuous increasing function of η in (α, β). Hence

$$\eta = \Psi^{-1}(u) \tag{19}$$

is a continuous increasing function of u in $[\Psi(\alpha), \Psi(\beta)]$, that is, by (15), in $[\Phi(a), \Phi(c)]$. Hence

$$\eta = \Psi^{-1}[\Phi(\xi)] = F(\xi) \tag{20}$$

is a continuous increasing function of ξ in (a, c).

This theorem suggests that although a cause X may be subject to some generally recognized frequency law $\varphi(x)$, its effect Y may be subject to almost any imaginable law $\psi(y)$, in the absence of rather definite knowledge of the functional relation between X and Y. In this connection, not only would $F(X)$ be available, as a possible expression of this unknown relation, but an infinite number of functions $f(X)$ to which Theorem I applies.

4. *Conclusion.* If the conditions of both theorems are satisfied everywhere, then there exists one and only one continuous increasing function $F(X)$ whose frequency law is identical with that of $f(X)$,—the cumulative law $\Psi(\eta)$, indeed, being the same for $F(X)$ and for $f(X)$ if only a finite number of points need to be excepted. Thus $F(X)$ may be looked upon as the chief representative of a whole class of functions $f(X)$ associated with each other through $\varphi(x)$.

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