A THEOREM ON SIMPLE ALGEBRAS*

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In a previous paper† I showed that every simple algebra A can be expressed as the direct product of a division algebra D and a simple matric algebra $M = (e_{pq})$; the object of this note is to show that this expression is unique, that is, *if* $A = D_1 \times M_1 = D_2 \times M_2$, *where* D_1 *and* D_2 *are division algebras and* M_1 *and* M_2 *are simple matric algebras, then* D_1 *and* M_1 *are simply isomorphic*‡ *with* D_2 *and* M_2 *respectively.*

Let δ_1 and δ_2 be the orders of D_1 and D_2, and let e_1 and e_2 be primitive idempotent elements of M_1 and M_2 respectively. If e_1 and e_2 are supplementary or equal, then§

$$D_1 \cong e_1 A e_1 \cong e_2 A e_2 \cong D_2\,;$$

M_1 and M_2 are then of the same order and are therefore simply isomorphic. We shall therefore suppose that $e_1 \neq e_2$ and, say, $e_1 e_2 \neq 0$.

Assume in the first place that $x = e_1 e_2$ is not nilpotent; there then exists a rational polynomial

$$y = \alpha_1 x + \alpha_2 x^2 + \cdots + \alpha_r x^r,$$

which is an idempotent element of the algebra X generated by x. Now $e_1 x = x$; therefore, since every element of X has the form $x f(x)$, $f(x)$ a polynomial in x, it follows that $e_1 y = y$ and

$$(y e_1)^2 = y e_1 y e_1 = y^2 e_1 = y e_1 = e_1 y e_1,$$

so that $y e_1 \leqq e_1 A e_1$; also $y e_1 \neq 0$ since $y e_1 y = y^2 = y \neq 0$; hence $y e_1$, being idempotent, equals e_1. In the same way

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† PROCEEDINGS OF THE LONDON SOCIETY, (2), vol. 6 (1907), p. 99.

‡ *Simple isomorphism* will be denoted by $\cong$.

§ See L. E. Dickson, *Algebras and Their Arithmetics*, Chicago, 1923, pp. 74, 77.

it follows that $e_2y = e_2$ and $ye_2 = y$. Also, if we set $y_{10} = y - e_1$, $y_{02} = y - e_2$, then

$$e_1y_{10} = y_{10},\ y_{10}e_1 = 0,\ e_2y_{02} = 0,\ y_{02}e_2 = y_{02},$$

and, since $e_1 \neq e_2$, one of y_{10}, y_{02} is not zero, say y_{10}.

We shall now show that y is primitive in A. Since $e_1y = y$, every element of yAy has the form*

$$w = d_{11}e_{11} + d_{12}e_{12} + \cdots, \quad (d_{ij} < D_1;\ e_{11} = e_1,\ e_{ij} < M_1).$$

If w is idempotent, this gives $d_{11} = a$, the modulus of A, so that, if w is primitive, $y - w$ lacks the term in e_{11} and, being therefore nilpotent, must equal 0 since $(y-w)^2 = y - w$. Hence y is primitive and we may set† $A = D \times M$ where M is a simple matric algebra which contains y and D is a division algebra simply isomorphic with yAy.

Now, since $ye_1 = e_1$,

$$De_1 = Dye_1 = yAye_1 \geqq ye_1Ae_1ye_1 = e_1Ae_1 = D_1e_1$$

also

$$D_1e_1 = e_1Ae_1 \geqq e_1yAye_1 = yAye_1 = Dye_1 = De_1;$$

hence $De_1 = D_1e_1$ and therefore $D \cong D_1$. Similarly $D \cong D_2$; hence $D_1 \cong D_2$ and, as before, also $M_1 \cong M_2$.

Suppose in the second place that e_1e_2 is nilpotent; then $(e_1e_2e_1)^r = (e_1e_2)^re_1$ is also nilpotent and, as e_1Ae_1 is a division algebra, it follows that $e_1e_2e_1 = 0$; hence $(e_2e_1)^2 = 0$ and so, by a repetition of the same argument, $e_2e_1e_2 = 0$. If now we set $y = e_1 - e_1e_2 - e_2e_1$, then

$$e_1y = e_1 - e_1e_2 \neq e_1,\ ye_1 = e_1 - e_2e_1,\ e_1ye_1 = e_1,$$
$$e_2y = 0 = ye_2,\ ye_1y = y,\ y^2 = y.$$

As before we must show that y is primitive. If $y = y_1 + y_2$ where y_1 and y_2 are supplementary idempotent elements, then

$$0 = e_2y = e_2y_1 + e_2y_2,$$

so that

$$0 = (e_2y_1 + e_2y_2)y_1 = e_2y_1,$$

and similarly

$$y_1e_2 = e_2y_2 = y_2e_2 = 0.$$

* See Dickson, loc. cit.

† See Dickson, loc. cit.

From $y_1 + y_2 = y = e_1 - e_1e_2 - e_2e_1$ we have therefore

$$y_1 = yy_1 = e_1y_1 - e_2e_1y_1,$$

so that $y_1e_1y_1 = y_1$, and similarly

$$y_2e_1y_2 = y_2, \ y_2e_1y_1 = y_2y_1 = 0 = y_1e_1y_2.$$

Now, if $z_1 = e_1y_1e_1$, $z_2 = e_1y_2e_1$, then

$$e_1 = e_1ye_1 = e_1y_1e_1 + e_1y_2e_1 = z_1 + z_2,$$

$$z_1^2 = e_1y_1e_1 \cdot e_1y_1e_1 = e_1 \cdot y_1e_1y_1 \cdot e_1 = e_1y_1e_1 = z_1,$$

$$z_1z_2 = e_1y_1e_1 \cdot e_1y_2e_1 = e_1y_1e_1y_2e_1 = 0,$$

and similarly $z_2^2 = z_2$, $z_2z_1 = 0$. Also $z_1 \neq 0$ unless $y_1 = 0$ since

$$y_1z_1y_1 = y_1e_1y_1e_1y_1 = y_1e_1y_1 = y_1,$$

and similarly $z_2 \neq 0$ unless $y_2 = 0$. But z_1 and z_2, if not zero, are supplementary idempotent elements in e_1Ae_1 whereas e_1 is primitive; hence one of them is zero, that is, y is primitive.

We may now, as before, set $A = D \times M$ where $D \cong yAy$ is a division algebra and M is a simple matric algebra containing y. Remembering that $e_1ye_1 = e_1$, we then have

$$e_1Dye_1 = e_1yAye_1 \geqq e_1ye_1Ae_1ye_1 = e_1Ae_1 = D_1e_1,$$

also

$$D_1e_1 = e_1Ae_1 \geqq e_1yAye_1 = e_1Dye_1;$$

hence $D_1e_1 = e_1Dye_1$. If d and d' are any elements of D, we therefore have

$$e_1dye_1 = d_1e_1, \ e_1d'ye_1 = d_1'e_1, \qquad (d_1, d_1' < D_1)$$

and hence

$$e_1(d + d')ye_1 = (d_1 + d_1')e_1,$$

$$d_1d_1'e_1 = e_1dye_1d'ye_1 = e_1dye_1yd'e_1 = e_1dyd'e_1 = e_1dd'ye_1,$$

and, since $e_1 = e_1ye_1$, it follows that $D \cong D_1$ and therefore $M \cong M_1$. Finally, since $e_2y = 0 = ye_2$, e_2 and y are supplementary and hence $D_2 \cong e_2Ae_2 \cong yAy \cong D \cong D_1$, from which, as before, $M_1 \cong M_2$. The proof of the theorem is therefore complete.

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