CONCERNING RELATIVELY UNIFORM CONVERGENCE*

BY R. L. MOORE

According to E. H. Moore, a sequence of functions $f_1(p)$, $f_2(p), f_3(p), \cdots$, defined on a range K, is said to converge, to a function $f(p)$, relatively uniformly with respect to the scale function $s(p)$ if, for every positive number e, there exists a positive number δ_e such that if $n > \delta_e$ then, for every p which belongs to K, $|f_n(p)-f(p)| < e|s(p)|$†.

In this note I will establish the following theorem.

THEOREM. *If S is a convergent sequence of measurable functions $f_1(x), f_2(x), f_3(x), \cdots$ defined on a measurable point set E and S converges for each x belonging to E, then E contains a subset E_0 of measure zero such that the sequence S converges relatively uniformly for all values of x on the range $E-E_0$.*

PROOF. Suppose that S converges on E to the limit function $f(x)$. By a theorem due to Egoroff‡, E contains a subset E_1 of measure less than 1 such that S converges to $f(x)$ uniformly on $E-E_1$. Similarly E_1 contains a subset E_2 of measure less than $1/2$ such that S converges to $f(x)$ uniformly on E_1-E_2. Continue this process thus obtaining a sequence of point sets $E_1, E_2, E_3, \cdots$ such that, for each n, (1) the measure of E_n is less than $1/n$, (2) E_{n+1} is a subset of E_n, (3) S converges uniformly on E_n-E_{n+1}. Let E_0 denote the set of points common to the sets $E_1, E_2, E_3, \cdots$. The set E_0 is either vacuous or of measure 0. Furthermore

$$E = E_0 + (E-E_1) + (E_1-E_2) + \cdots .$$

Since S converges uniformly on each point set of the countable collection $E-E_1$, E_1-E_2, $E_2-E_3, \cdots$, it

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† See E. H. Moore, *Introduction to a Form of General Analysis*, The New Haven Mathematical Colloquium (Yale University Press, New Haven, 1910).

‡ COMPTES RENDUS, Jan. 30, 1911.

follows, by a theorem due to E. W. Chittenden, that* S converges relatively uniformly on the sum of all the point sets of this collection. But this sum is $E - E_0$.

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THE THEORY OF CLOSURE OF TCHEBYCHEFF POLYNOMIALS FOR AN INFINITE INTERVAL†

BY J. A. SHOHAT (J. CHOKHATE)

1. *The Theorem of Closure.* Suppose we have a function $p(x)$, not negative in a given interval (a, b), for which all the integrals

$$\int_a^b p(x) x^n dx, \qquad (n = 0, 1, 2, \cdots)$$

exist. It is well known that we can form a normal and orthogonal system of polynomials

$$\varphi_n(x) = a_n x^n + \cdots, \quad a_n > 0, \quad (n = 0, 1, 2, \cdots),$$

uniquely determined by means of the relations

$$\int_a^b p(x) \varphi_m(x) \varphi_n(x) dx = \begin{cases} 0, & m \neq n, \\ 1, & m = n. \end{cases}$$

We call these polynomials *Tchebycheff polynomials* corresponding to the interval (a, b) with the *characteristic function* $p(x)$. The simplest example is given by Legendre polynomials, corresponding to the interval $(-1, +1)$ with $p(x) = 1$.

The most important application of Tchebycheff polynomials is their use in the development of functions into

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* E. W. Chittenden, *Relatively uniform convergence of sequences of functions,* TRANSACTIONS OF THIS SOCIETY, vol. 15 (1914), pp. 197–201. As Chittenden observes, this is an extension of a theorem given by E. H. Moore on page 87 of his *Introduction to a Form of General Analysis,* loc. cit.

† Presented to the Society, December 29, 1923.