The relation (9) follows by comparing (6) with the identity

$$\vartheta_0\vartheta_3^2 = \left[1 + 2\sum_{a=1}^{\infty} q^{a^2}(-1)^a\right]\left[1 + 4\sum_{n=1}^{\infty} q^n \xi(n)\right],$$

the second factor on the right being the algebraic expression of the well known theorem which gives the number of representations of n as a sum of two integer squares.

THE UNIVERSITY OF WASHINGTON

NOTE ON A SPECIAL CONGRUENCE*

BY MALCOLM FOSTER

1. *Introduction.* Let S be any surface referred to its lines of curvature. With every point M of S we associate the trihedral of the surface, taking the x-axis of the trihedral tangent to the curve $v =$ const. We consider the congruence of lines l parallel to the z-axis, the normal to S, which pierce the xy-plane at the point $(\xi, \eta_1, 0)$.† The equations of l are $x = \xi$, $y = \eta_1$, and the coordinates of any point on l are

(1) $$x = \xi, \quad y = \eta_1, \quad z = t,$$

where t is the distance on l measured from the point $(\xi, \eta_1, 0)$.

2. *Condition for a Normal Congruence.* If there be a surface normal to the congruence we must have

(2) $$\delta z = dz + p_1 y dv - q x du = 0,‡$$

for all values of $\frac{dv}{du}$. Using (1) equation (2) becomes

(3) $$\frac{\partial t}{\partial u} du + \frac{\partial t}{\partial v} dv + p_1 \eta_1 dv - q\xi du \equiv 0;$$

hence

(4) $$\frac{\partial t}{\partial u} - q\xi = 0, \qquad \frac{\partial t}{\partial v} + p_1\eta_1 = 0.$$

---

* Presented to the Society, May 3, 1924.

† The notation used in this paper is the same as in Eisenhart's *Differential Geometry of Curves and Surfaces*; see pp. 166–176.

‡ Eisenhart, p. 170.

The condition of integrability is

$$\frac{\partial}{\partial v}\left(\frac{\partial t}{\partial u}\right) = \frac{\partial}{\partial u}\left(\frac{\partial t}{\partial v}\right).$$

Hence from (4) we must have

$$q\frac{\partial \xi}{\partial v} + \xi\frac{\partial q}{\partial v} = -\left(p_1\frac{\partial \eta_1}{\partial u} + \eta_1\frac{\partial p_1}{\partial u}\right),$$

which reduces to* $q\eta_1 r - p_1\xi r - p_1\xi r_1 + q\eta_1 r_1 = 0$. This may be written in the form

$$(5) \qquad (q\eta_1 - p_1\xi)(r + r_1) = 0.$$

Conversely, when (5) is satisfied, the function t as given by (4) satisfies (3), and hence there exists a single infinity of parallel surfaces normal to the lines l. If $q\eta_1 - p_1\xi = 0$, the surface S is minimal.† Hence we have the following theorem.

THEOREM I. *A necessary and sufficient condition that the congruence of lines l be normal is that S be a minimal surface, or else that $r + r_1 = 0$.*

3. *Equation of the Curves defining the Developables.* As M, the vertex of the trihedral, is displaced along some curve C on S the locus of l is a ruled surface of the congruence; we seek the equation of the curves C for which the locus of l is developable. For l to generate a developable surface the displacement of some point on l must be in the direction of the line; hence for that point

$$(6) \qquad \delta x = \delta y = 0.$$

By means of (1), equations (6) take the forms‡

$$(7) \quad \begin{cases} \dfrac{\partial \xi}{\partial u}du - \eta_1 r dv + \xi du + tq du - \eta_1(r du + r_1 dv) = 0, \\ \xi r_1 du + \dfrac{\partial \eta_1}{\partial v}dv + \eta_1 dv - tp_1 dv + \xi(r du + r_1 dv) = 0. \end{cases}$$

The elimination of t between these two equations gives

* Eisenhart, p. 168, formulas (48), and p. 170, formulas (55).

† Eisenhart, p. 174, formulas (73).

‡ Eisenhart, p. 170.

the following equation of the curves on S defining the developable surfaces of the congruence:

$$(8)\qquad q\xi(r+r_1)du^2+\left[p_1\left(\frac{\partial\xi}{\partial u}+\xi-\eta_1 r\right)+q\left(\frac{\partial\eta_1}{\partial v}+\eta_1+\xi r_1\right)\right]du\,dv-p_1\eta_1(r+r_1)dv^2=0.$$

The elimination of the ratio $\frac{dv}{du}$ between equations (7) gives the following equation for the distances along l to the focal points:

$$(9)\qquad p_1qt^2+\left[p_1\left(\frac{\partial\xi}{\partial u}+\xi-\eta_1 r\right)-q\left(\frac{\partial\eta_1}{\partial v}+\eta_1+\xi r_1\right)\right]t-\left(\frac{\partial\xi}{\partial u}+\xi-\eta_1 r\right)\left(\frac{\partial\eta_1}{\partial v}+\eta_1+\xi r_1\right)-\xi\eta_1(r+r_1)^2=0.$$

The condition that equation (8) define an orthogonal system is*

$$\xi^2p_1\eta_1(r+r_1)-\eta_1^2q\xi(r+r_1)=0,$$

which may be written

$$(10)\qquad (q\eta_1-p_1\xi)(r+r_1)=0,$$

since $\xi,\eta_1 \neq 0$. Since (10) is identical with (5) we have the following theorem.

THEOREM II. *If the congruence of lines l be normal the developables are represented on S by an orthogonal system.*

We note from (8) that if the congruence be normal by virtue of the relation $r+r_1=0$ the curves defining the developables are the lines of curvature.

4. *Asymptotic Lines on the Normal Surfaces.* Let C be any curve on S through M, the vertex of the trihedral, and let l and l_1 be the lines of the congruence corresponding to M and M_1, a neighboring point on C. As M_1 approaches M along C the foot on l of the common perpendicular to l and l_1 approaches a certain limiting position called the central point of the generator l. The locus of the central points is the line of striction of the ruled surface

* Eisenhart, p. 80.

defined by C. We wish to determine the distance along l to the line of striction of this ruled surface.

To this end we consider a second trihedral T_0* with vertex at some fixed point in space, whose x_0-, y_0-, and z_0-axes are in every position parallel to the x-, y-, and z-axes of the moving trihedral. Relative to the trihedral T_0 the coordinates of the point on the unit sphere corresponding to l are (0, 0, 1). As the vertex of the moving trihedral is displaced along C the absolute displacements of the point (0, 0, 1) in the directions of the axes of the trihedral T_0 will be the variations experienced by the direction-cosines of l. If these variations are denoted by $\delta\alpha$, $\delta\beta$, $\delta\gamma$, we have† $\delta\alpha = qdu$, $\delta\beta = -p_1 dv$, $\delta\gamma = 0$, since for the motion of the trihedral T_0 the translations ξ and η_1 are zero. The direction-cosines of l_1 relative to the trihedral at M are therefore qdu, $-p_1 dv$, 1. The displacement of the central point on l must be orthogonal to both l and l_1. Hence for that point we must have $\delta z = 0$, $qdu\delta x - p_1 dv \delta y + \delta z = 0$. Combining these equations we get $qdu\delta x - p_1 dv\delta y = 0$, which becomes

$$qdu\left[\frac{\partial \xi}{\partial u}du - \eta_1 r dv + \xi du + qtdu - \eta_1 r du - \eta_1 r_1 dv\right]$$
$$-p_1 dv\left[\xi r_1 du + \frac{\partial \eta_1}{\partial v}dv + \eta_1 dv + \xi r du + \xi r_1 dv - tp_1 dv\right] = 0.$$

This may be written in the form

$$(11)\quad q\left(\frac{\partial \xi}{\partial u} + \xi + qt - \eta_1 r\right)du^2 - (q\eta_1 + p_1\xi)(r + r_1)dudv$$
$$-p_1\left(\frac{\partial \eta_1}{\partial v} + \eta_1 + \xi r_1 - tp_1\right)dv^2 = 0.$$

When the value of dv/du which determines the curve C is put in this equation we have an equation in t which determines the distance along l to the line of striction of the ruled surface defined by C. Conversely, given t a function of u

* Eisenhart, p. 168.

† Eisenhart, p. 170.

and v, the equation (11) determines two curves on S, though not necessarily real, which define two ruled surfaces of the congruence for which t is the distance to their lines of striction.

We suppose that the congruence is normal by virtue of the relation $r+r_1=0$. Hence equation (11) is of the form

$$L(u,v)\,du^2-M(u,v)\,dv^2=0,$$

and consequently represents a system of curves symmetrically placed relative to the lines of curvature. Now the normals to a surface along the asymptotic lines form ruled surfaces for which the asymptotic lines are the lines of striction.* Hence we have the following theorem.

THEOREM III. *If the congruence of lines l be normal by virtue of the relation $r+r_1=0$, the curves on S which represent the asymptotic lines on the normal surfaces form a system which is symmetrically placed relative to the lines of curvature.*

5. *Minimal Surfaces.* We now suppose that S is a minimal surface with the parameters of the lines of curvature so chosen that the linear element of the surface has the form†

$$ds^2=\varrho(du^2+dv^2), \tag{12}$$

where ϱ is the absolute value of each principal radius. Hence we have‡

$$\left\{\begin{aligned}
&\xi=\eta_1=\sqrt{\varrho},\quad q=-\frac{D}{\sqrt{E}}=-\frac{1}{\sqrt{\varrho}},\\
&p_1=\frac{D''}{\sqrt{G}}=-\frac{1}{\sqrt{\varrho}},\\
&r=-\frac{1}{\sqrt{G}}\frac{\partial}{\partial v}\sqrt{E}=-\frac{\frac{\partial\varrho}{\partial v}}{2\varrho},\\
&r_1=\frac{1}{\sqrt{E}}\frac{\partial}{\partial u}\sqrt{G}=\frac{\frac{\partial\varrho}{\partial u}}{2\varrho}.
\end{aligned}\right. \tag{13}$$

---

* Eisenhart, p. 268, Ex. 14.

† Eisenhart, p. 253.

‡ Eisenhart, p. 174.

When the values of ξ, η_1, q, p_1, r, r_1, as given in (13) are substituted in (9), it is readily seen that the coefficient of t vanishes. Hence we have the following theorem.

THEOREM IV. *If S be a minimal surface with the parameters of the lines of curvature so chosen that the linear element has the form* (12), *the congruence of lines l is normal and has for its middle envelope the given minimal surface.*

The minimal surface S is therefore the mean evolute of each of the normal surfaces.

6. *Envelope of a Two-Parameter Family of Surfaces.* Let S be any surface referred to any parametric system. With every point M of S we associate the trihedral of the surface, giving the x-axis its most general orientation relative to the curve $v =$ const. Let

$$F(x, y, z, u, v) = 0 \tag{14}$$

be the equation relative to the trihedral at M of some surface $\sum$. We consider the envelope of such a two-parameter family of surfaces.

The characteristic is defined by (14) and the equations

$$\text{(15)}\quad \begin{cases} \dfrac{\partial F}{\partial x}\dfrac{dx}{du}+\dfrac{\partial F}{\partial y}\dfrac{dy}{du}+\dfrac{\partial F}{\partial z}\dfrac{dz}{du}+\dfrac{\partial F}{\partial u}=0, \\[2ex] \dfrac{\partial F}{\partial x}\dfrac{dx}{dv}+\dfrac{\partial F}{\partial y}\dfrac{dy}{dv}+\dfrac{\partial F}{\partial z}\dfrac{dz}{dv}+\dfrac{\partial F}{\partial v}=0. \end{cases}$$

Since the characteristic is fixed in space, we must have

$$\begin{aligned} \delta x &= dx+\xi du+\xi_1 dv+(q du+q_1 dv)z-(r du+r_1 dv)y=0, \\ \delta y &= dy+\eta du+\eta_1 dv+(r du+r_1 dv)x-(p du+p_1 dv)z=0, \\ \delta z &= dz \qquad\qquad\qquad +(p du+p_1 dv)y-(q du+q_1 dv)x=0, \end{aligned}$$

for all values of $\dfrac{dv}{du}$. Hence

$$\text{(16)}\quad \begin{cases} \dfrac{dx}{du}=ry-qz-\xi,\ \dfrac{dy}{du}=pz-rx-\eta,\ \dfrac{dz}{du}=qx-py, \\[2ex] \dfrac{dx}{dv}=r_1y-q_1z-\xi_1,\dfrac{dy}{dv}=p_1z-r_1x-\eta_1,\dfrac{dz}{dv}=q_1x-p_1y. \end{cases}$$

By (16), equations (15) become

$$
(17) \quad \begin{cases} \dfrac{\partial F}{\partial x}(ry - qz - \xi) + \dfrac{\partial F}{\partial y}(pz - rx - \eta) \\ \qquad\qquad + \dfrac{\partial F}{\partial z}(qx - py) + \dfrac{\partial F}{\partial u} = 0, \\ \dfrac{\partial F}{\partial x}(r_1 y - q_1 z - \xi_1) + \dfrac{\partial F}{\partial y}(p_1 z - r_1 x - \eta_1) \\ \qquad\qquad + \dfrac{\partial F}{\partial z}(q_1 x - p_1 y) + \dfrac{\partial F}{\partial v} = 0. \end{cases}
$$

The coordinates $(x, y, z,)$ of the characteristic are therefore given by equations (14) and (17).

7. *Applications.* As an application, consider the envelope of certain two-parameter families of planes. We choose S as any surface referred to its lines of curvature, and choose the x-axis of the trihedral tangent to the curve $v = \text{const.}$ Consider the two two-parameter families of planes

$$
(18) \qquad x = \xi,
$$

and

$$
(19) \qquad y = \eta_1.
$$

For the family of planes (18), equations (17) become

$$
(20) \qquad ry - qz - \xi - \frac{\partial \xi}{\partial u} = 0, \quad r_1 y + \eta_1 r = 0.
$$

Hence solving equations (18) and (20) the coordinates of the characteristic of the planes (18) are

$$
(21) \quad x_1 = \xi, \quad y_1 = -\frac{\eta_1 r}{r_1}, \quad z_1 = -\frac{r_1\left(\xi + \dfrac{\partial \xi}{\partial u}\right) + \eta_1 r^2}{q r_1}.
$$

For the family of planes (19), equations (17) become

$$
(22) \qquad rx + \xi r_1 = 0, \quad p_1 z - r_1 x - \eta_1 - \frac{\partial \eta_1}{\partial v} = 0.
$$

The solution of equations (19) and (22) gives the following for the coordinates of the characteristic of the planes (19):

$$
(23) \quad x_2 = -\frac{\xi r_1}{r}, \quad y_2 = \eta_1, \quad z_2 = \frac{r\left(\eta_1 + \dfrac{\partial \eta_1}{\partial v}\right) - \xi r_1^2}{r p_1}.
$$

Let us now assume the relation

$$r + r_1 = 0. \tag{24}$$

Then, from (21) and (23), we have

$$\left\{\begin{aligned} &x_1 = x_2 = \xi, \qquad y_1 = y_2 = \eta_1, \\ &z_1 = \frac{\eta_1 r - \xi - \dfrac{\partial \xi}{\partial u}}{q}, \qquad z_2 = \frac{\xi r_1 + \eta_1 + \dfrac{\partial \eta_1}{\partial v}}{p_1}. \end{aligned}\right. \tag{25}$$

Hence when the relation (24) holds, the characteristics of both families of planes lie on the line l. Moreover, when the relation (24) holds the roots of equation (9) are

$$t_1 = \frac{\eta_1 r - \xi - \dfrac{\partial \xi}{\partial u}}{q}, \qquad t_2 = \frac{\xi r_1 + \eta_1 + \dfrac{\partial \eta_1}{\partial v}}{p_1}. \tag{26}$$

Hence, since t_1 and t_2 are identical with z_1 and z_2 in (25) we have the following theorem.

THEOREM V. *If S be a surface referred to its lines of curvature for which the relation $r + r_1 = 0$ holds, the envelopes of the two families of planes $x = \xi$ and $y = \eta_1$ are the two focal sheets of the normal congruence of lines l.*

The planes (18) and (19) are therefore the focal planes of the congruence. From (8), the curves defining the developables are the lines of curvature. Hence from (7),

$$t = t_1 = \frac{\eta_1 r - \xi - \dfrac{\partial \xi}{\partial u}}{q},$$

for $v = \text{const.}$, and

$$t = t_2 = \frac{\xi r_1 + \eta_1 + \dfrac{\partial \eta_1}{\partial v}}{p_1},$$

for $u = \text{const.}$ Consequently the planes $x = \xi$ and $y = \eta_1$ envelope the focal sheets determined by the developables $v = \text{const.}$ and $u = \text{const.}$, respectively.

YALE UNIVERSITY