COMPLETE CLASS NUMBER EXPANSIONS FOR CERTAIN ELLIPTIC THETA CONSTANTS OF THE THIRD DEGREE*

BY E. T. BELL

1. *Introduction and Summary.* The expansions of the nine possible theta constants of the third degree in which the parameter of each theta is q,

$$\vartheta_0^3,\ \vartheta_2^3,\ \vartheta_3^3,\ \vartheta_0\vartheta_2^2,\ \vartheta_0\vartheta_3^2,\ \vartheta_2\vartheta_3^2,\ \vartheta_0^2\vartheta_2,\ \vartheta_0^2\vartheta_3,\ \vartheta_2^2\vartheta_3,$$

are of fundamental importance for the arithmetic of class number relations. Two of these. (6), (7) below, have not previously been stated. The expansions of this pair depend upon those of the remaining seven and upon some additional facts concerning representations in a certain ternary quadratic form. The present paper therefore contains the complete set of expansions.

In a recent note† the author has shown that the classical series of Kronecker and Hermite, in which all summations are with respect to $n = 0$ to ∞,

$$(1)\qquad \begin{aligned}\vartheta_2(q^4)\vartheta_3^2(q^4) &= 4\sum q^{4n+1}F(4n+1),\\ \vartheta_2^2(q^4)\vartheta_3(q^4) &= 4\sum q^{4n+2}F(4n+2),\end{aligned}$$

are immediate consequences of the algebraic equivalent

$$(2)\qquad \vartheta_3^3 = 12\sum q^n E(n),\quad E(n) \equiv F(n)-F_1(n),$$

of the theorem of Gauss on representations as sums of three squares.

It was remarked also that their series

$$(3)\qquad \vartheta_2^3(q^4) = 8\sum q^{8n+3}F(8n+3)$$

is also implied by another result due to Gauss. In the foregoing, $F(n)$, $F_1(n)$ denote, respectively, the numbers of

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† This BULLETIN, vol. 30, p. 236.

odd, even classes, with all the usual conventions, of binary quadratic forms of negative determinant $-n$, so that $E(0) = 1/12$.

From (1) on replacing q by $\sqrt[4]{q}$, changing q into $-q$ in the result, and then replacing q by q^4, we get

$$(4)\qquad \begin{aligned} \vartheta_0^2(q^4)\vartheta_2(q^4) &= 4\sum q^{4n+1}(-1)^n F(4n+1), \\ \vartheta_0(q^4)\vartheta_2^2(q^4) &= 4\sum q^{4n+2}(-1)^n F(4n+2), \end{aligned}$$

and from (2) by changing the sign of q,

$$(5)\qquad \vartheta_0^3 = 12\sum q^n(-1)^n E(n).$$

There thus remain to be expanded only $\vartheta_0\vartheta_3^2$ and $\vartheta_0^2\vartheta_3$. These expansions are of equal importance with the rest. For example, the series for these two constants are needed more frequently than those for any of the others when we attempt to elicit from Jacobi's theta formula (or from the three-term equations of Weierstrass, Kronecker, or Briot and Bouquet) the numerous class number relations containing arbitrary odd or even functions which such formulas imply. The expansions do not seem to have been obtained hitherto. We shall show that

$$(6)\qquad \vartheta_0\vartheta_3^2 = 4\sum q^n\Big[(-1)^n + \{1+(-1)^n\}\cos n\frac{\pi}{2} + \{1-(-1)^n\}\sin n\frac{\pi}{2}\Big]E(n),$$

and therefore, on changing q into $-q$ and at the same time indicating the corresponding alternative form for (6), we have

$$(7)\qquad \vartheta_0^2\vartheta_3 = 4\sum q^n[1+i^n(1+i)\{1-(-1)^n i\}]E(n), \quad i=\sqrt{-1}.$$

The structure of the coefficient $C(n)$ of q^n in (6) may be noted. It is readily seen that

$$(8.1)\qquad C(4n) = 12E(4n) = 12E(n),$$

$$(8.2)\qquad C(4n+1) = 4E(4n+1) = 4F(4n+1),$$

$$(8.3)\qquad C(4n+2) = -4E(4n+2) = -4F(4n+2),$$

$$(8.4)\qquad C(8n+3) = -12E(8n+3) = -8F(8n+3),$$

$$(8.5)\qquad C(8n+7) = -12E(8n+7) = 0,$$

the third members of which follow from the well known elementary reduction formulas for $E(n)$.

To check (6), (7) we shall write down for $n>0$ an interesting consequence of (6),

$$(9)\quad 2(-1)^n\epsilon(n)+4\xi(n)+8\sum(-1)^a\xi(n-a^2) = C(n),$$

in which $\epsilon(n)=1$ or 0 according as n is or is not the square of a positive integer, $\xi(n)=$ the excess of the number of divisors of n that are $\equiv 1 \bmod 4$ over the number of those $\equiv -1 \bmod 4$, and the summation refers to all $a=1, 2, 3, \ldots,$ such that $n-a^2>0$. I have verified (9) numerically, hence checking (6), (7).

2. *Expansion of* $\vartheta_0\vartheta_3^2$. Write $N(n=f)$ for the number of representations of the integer $n\geqq 0$ in the form f. Then

$$(10)\quad \vartheta_0\vartheta_3^2 = \sum q^n\big[N(n = 4n_1^2+n_2^2+n_3^2) \\ -N(n = m_1^2+n_2^2+n_3^2)\big],$$

in which n_1, n_2, n_3 are integers $\gtreqless 0$, and $m_1 \gtrless 0$ is an odd integer. From (1), (2) we have

$$N(4n+1 = m_1^2+4n_2^2+4n_3^2) = 4F(4n+1) = 4E(4n+1),$$
$$N(4n+2 = m_1^2+m_2^2+4n_3^2) = 4F(4n+2) = 4E(4n+2),$$
$$N(4n+3 = m_1^2+m_2^2+m_3^2) = 12E(8n+3),$$

and we transcribe from a former paper,*

$$N(4n = 4n_1^2+n_2^2+n_3^2) = 12E(n) = 12E(4n),$$
$$N(4n+1 = 4n_1^2+n_2^2+n_3^2) = 8F(4n+1) = 8E(4n+1),$$
$$N(4n+2 = 4n_1^2+n_2^2+n_3^2) = 4F(4n+2) = 4E(4n+2),$$
$$N(4n+3 = 4n_1^2+n_2^2+n_3^2) = 0.$$

From these and (10), since in (10) either of n_2, n_3 may be odd or even, it follows on referring to (8.1)-(8.5) that

$$\vartheta_0\vartheta_3^2 = \sum q^nC(n),$$

which is equivalent to (6).

* American Mathematical Monthly, vol. 21 (1924), p. 126.

The relation (9) follows by comparing (6) with the identity

$$\vartheta_0\vartheta_3^2 = \left[1 + 2\sum_{a=1}^{\infty} q^{a^2}(-1)^a\right]\left[1 + 4\sum_{n=1}^{\infty} q^n \xi(n)\right],$$

the second factor on the right being the algebraic expression of the well known theorem which gives the number of representations of n as a sum of two integer squares.

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NOTE ON A SPECIAL CONGRUENCE*

BY MALCOLM FOSTER

1. *Introduction.* Let S be any surface referred to its lines of curvature. With every point M of S we associate the trihedral of the surface, taking the x-axis of the trihedral tangent to the curve $v =$ const. We consider the congruence of lines l parallel to the z-axis, the normal to S, which pierce the xy-plane at the point $(\xi, \eta_1, 0)$.† The equations of l are $x = \xi$, $y = \eta_1$, and the coordinates of any point on l are

(1) $$x = \xi, \quad y = \eta_1, \quad z = t,$$

where t is the distance on l measured from the point $(\xi, \eta_1, 0)$.

2. *Condition for a Normal Congruence.* If there be a surface normal to the congruence we must have

(2) $$\delta z = dz + p_1 y dv - q x du = 0,‡$$

for all values of $\frac{dv}{du}$. Using (1) equation (2) becomes

(3) $$\frac{\partial t}{\partial u} du + \frac{\partial t}{\partial v} dv + p_1 \eta_1 dv - q\xi du \equiv 0;$$

hence

(4) $$\frac{\partial t}{\partial u} - q\xi = 0, \qquad \frac{\partial t}{\partial v} + p_1 \eta_1 = 0.$$

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† The notation used in this paper is the same as in Eisenhart's *Differential Geometry of Curves and Surfaces*; see pp. 166–176.

‡ Eisenhart, p. 170.