# GOURSAT ON PFAFF'S PROBLEM

*Leçons sur le Problème de Pfaff.* By Edouard Goursat, Paris, J. Hermann, 1922. 386 pp.

On the first page of Volume 3 of the *Institutiones Calculi Integralis,* Euler declares with reference to the equation $Pdx + Qdy + Rdz = 0$, that there must be some multiplier $M$ by means of which the expression on its left-hand side will become an exact differential; "for unless such a multiplier existed, the proposed differential equation would become absurd and would mean nothing at all". If this opinion had remained unchallenged, the volume under review would not have had much reason for existence. For the problem of Pfaff with which it is occupied relates to differential expressions of which the linear form $\sum_{i=1}^{n} X_i\,(x_1, \cdots, x_n)\,dx_i$ is the simplest type, independently of whether or not there exists a multiplier which renders them exact.

In a memoir of 1814, J. F. Pfaff showed the essential equivalence of the partial differential equation of the first order

$$\frac{dz}{dx_1} = \psi\left((z, x_1, \cdots, x_n, \frac{dz}{dx_2}, \cdots, \frac{dz}{dx_n}\right)$$

and of the total differential equation

$$dz - \psi(z, x_1, \cdots, x_n, p_2, \cdots, p_n)dx_1 - p_2dx_2 - \cdots - p_ndx_n = 0.$$

A solution of the latter equation consists of a set of $n$ equations in $z, x_1, \cdots, x_n, p_2, \cdots, p_n$, which are solvable for $z, p_2, \cdots, p_n$ in terms of $x_1, \cdots, x_n$ and thus yield functions which reduce this equation to an identity in $x_1, \cdots, x_n$ and their differentials. It is a special case of the general equation $X_1dx_1 + \cdots + X_ndx_n = 0$, in which $X_i$ are functions of $x_1, \cdots, x_n$. The reduction of this equation and of the differential form on its left-hand side has been the object of many writers, among whom are Gauss, Jacobi, Grassmann, Clebsch, Darboux, Frobenius. This question occupies nearly all of Part I of Forsyth's *Theory of Differential Equations,* to which M. Goursat refers for a historical account of the development of the subject. It also furnishes the material for E. vonWeber's *Vorlesungen über das Pfaffsche Problem,* to which only very casual reference is made in M. Goursat's volume. Indeed the latter is very different in character, for it is not primarily concerned with an account of the methods for the solution of the problem developed by various writers, but rather with its extensions and generalizations. This indeed is its real reason for existence as a separate volume.

The first two chapters form a natural sequel to the author's *Leçons* *sur l'Intégration des Équations aux Dérivées Partielles du Premier Ordre* to which they are similar in style. In the first, we find a thorough treatment of the fundamental theorem of Pfaff on the reduction of the linear differential form $\sum^{n} X_i dx_i$ to one or the other of the canonical forms $z_1 dy_1 + \cdots + z_p dy_p + dy_{p+1}$ or $z_1 dy_1 + \cdots + z_p dy_p$ according as the class of the form is $2p + 1$ or $2p$, where $z_1, \cdots, z_p, y_1, \cdots, y_{p+1}$ are linearly independent functions of $x_1, \cdots, x_n$. The concept "class of a form" introduced by Frobenius and the reductions to the canonical forms are developed here by the very elegant methods of Frobenius and Darboux, which are based on a consideration of a bilinear covariant of $\omega$, viz.

$$\omega' = \sum a_{ij} dx_i \delta x_j,$$

where

$$a_{ij} = \frac{\partial X_i}{\partial x_j} - \frac{\partial X_j}{\partial x_i},$$

and of four associated covariant linear differential systems. The matrix of the $a_{ij}$ being skew symmetric, it becomes apparent that the even- or oddness of the class $c$ will be of importance. This distinction does indeed run throughout the whole theory of the forms of Pfaff. A good knowledge of completely integrable linear differential systems is necessary to understand this part of the book.

Chapter 2 is devoted to the problem from which the book derives its title, viz., that of solving the equation $\omega = 0$. A solution consists of $p$ equations $f_i(x_1, \cdots, x_n) = 0$, $i = 1, \cdots, p$, $p < n$, such that from them together with those obtained from them by differentiation, the given equation follows. It is shown that the maximum number of dimensions of a manifold which in this sense satisfies the equation is $n - (\gamma + 1)/2$, where $\gamma$ is the class of the equation, a number which is equal to $c$ or to $c - 1$, according as $c$ is odd or even; every integral manifold of lower dimensionality lies in this maximum-dimensional manifold; moreover, the maximum number of dimensions is never less than the integral part of $n/2$. The families of characteristic manifolds are determined by the $\gamma$ solutions of one of the associated complete linear differential systems referred to above, and a general theorem is obtained establishing their relation to the integral manifolds.

The second division of the book consisting of Chapters 3, 4, and 5 deals with the generalization of the problem of Pfaff to differential forms of higher degree

$$\Omega \equiv \sum A_{\alpha_1 \ldots \alpha_p} \, dx_{\alpha_1} \ldots dx_{\alpha_p}$$

in which $dx_{\alpha_1} \ldots dx_{\alpha_p}$ is a symbolic product whose characteristic properties are to be determined. To justify this problem as an extension of the problem of Pfaff, the latter is restated in the following form: To determine in $n$-space all the $r$-manifolds $(1 \leqq r < n)$ such that the line integral $\int \omega$ shall vanish if taken along any curve on such a manifold. The generalized problem is then to determine in $n$-space all $r$-manifolds $p \leqq r < n$ such that the $\int \Omega$ vanishes if taken over any $p$-manifold situated on them. The theory of the symbolic product $dx_{\alpha_1} \ldots dx_{\alpha_p}$, and of the symbolic forms $\Omega$, as developed and applied in Chapters 3 and 4 is largely the work of MM. Cartan and Goursat. As remarked in the preface, the study of the symbolic products could readily be connected with the tensor calculus. It seems to the reviewer indeed a wise decision to have given the development independently of the general theory.

The bilinear covarient $\omega'$ introduced for the simple case by Darboux and Frobenius finds an extension in the derivative $\Omega'$ of the form $\Omega$. As in the linear problem the vanishing of $\omega'$ was the necessary and sufficient condition for exactness, so $\Omega' = 0$ is the necessary and sufficient condition for the exactness of $\Omega$. The concept "class of a form" is extended as are also the associated differential systems.

Of especial interest in connection with the results of Chapters 1 and 2 are the applications of the symbolic forms to the equation of Pfaff. The $p^{\text{th}}$ derivative of a linear form $\omega$ being a symbolic form of degree $p+1$, defined by the recursion formulas

$$\omega^{(2m-1)} = \frac{1}{m!}(\omega')^m,$$

$$\omega^{(2m)} = \omega(\omega')^m,$$

it is found that if the $c^{\text{th}}$ derivative is the first among the successive derivatives of a linear form $\omega$ which vanishes, then the class of $\omega$ is equal to $c$. The derivative forms furnish moreover in a very direct way a method for the reduction of a form of Pfaff to the canonical form discussed in Chapters 1 and 2. Chapter 4 closes with the application of these results to a system of partial differential equations and to the problem of contact transformations.

The fundamental properties of the integral invariants of a system of differential equations

$$\frac{dx_1}{X_1} = \frac{dx_2}{X_2} = \cdots = \frac{dx_n}{X_n} = dt$$

are shown in Chapter 5 to follow quite naturally from the properties of the symbolic forms considered in the preceding chapter. The treatment followed complements in an interesting manner that of Cartan in his recent book on Integral Invariants (see this BULLETIN, vol. 29, p. 140). There are a variety of beautiful results: the relation between the invariant character of $\int \omega$ and of $\int \omega'$, $\omega$ being a form of any degree; the theorem that every relative integral invariant is the sum of an absolute integral invariant and the integral of a symbolic total differential, etc.

The last three chapters are devoted to recent results in the study of systems of Pfaff, a great part of which is due to Cartan. The first question considered in Chapter 6 is as to the maximum number of dimensions $\rho$ of an integral manifold of a system of $r$ forms of Pfaff.

There is a rich amount of material, too varied to lend itself to brief discussion, on systems of linear differential expressions, contact transformations, derived systems, the problem of Monge, second order partial differential equations, etc. The final chapter is devoted to the classification of the integral elements of a system of Pfaff and to the existence theorem.

The book affords an excellent introduction to the study of a problem which occupies a very central position in the theory of differential equations. It gives a detailed survey of the classical theory, of its connections with other domains, of its most modern delelopments, and of the directions in which further advances may be made.

A. DRESDEN

# WATSON ON BESSEL FUNCTIONS

*A Treatise on the Theory of Bessel Functions.* By G. N. Watson. Cambridge, University Press, 1922. viii + 804 pages.

The purpose of this book is twofold: to develop certain applications of the fundamental processes of the theory of functions of complex variables for which Bessel functions are admirably adapted; and secondly, to compile a collection of results which shall be of value to the increasing number of mathematicians and physicists who encounter Bessel functions in the course of their researches. The author believes that the existence of such a collection is demanded by the greater abstruseness of properties of Bessel functions (especially of functions of large order) which have been required in recent years in various problems of mathematical physics.

In his exposition the author has endeavoured to accomplish two specific results: to give an account of the theory of Bessel functions which a pure mathematician would regard as fairly complete; and to include all formulas, whether general or special, which, although without theoretical interest, are likely to be required in practical applications. An attempt is made to give the latter results, as far as possible, in a form appropriate for use in the applications. These exalted aims the author seems to have achieved with a remarkable success. The