librium and of natural systems (whether physical or economic) and collated with the very general viewpoint of Royce and of C. S. Peirce (whose maturer work Keynes does not cite), might be worthy of at least a bibliographic reference by an author who is setting up a category of probability. However, it would be unreasonable to expect any discussion of categories to reach nearer the date of issue than about 50 years, just as one can hardly expect the full treatment of the necessary and sufficient conditions justifying a new analytical method to follow right on the heels of the introduction of such a method by the physicist (Fourier, Heaviside).

E. B. WILSON

BLASCHKE ON DIFFERENTIAL GEOMETRY

Vorlesungen über Differentialgeometrie und geometrische Grundlagen von Einstein's Relativitätstheorie. Volume 1, *Elementare Differentialgeometrie.* By W. Blaschke. Berlin, Julius Springer, 1921. x + 230 pp.

This volume is the first of a series of three which the author plans to publish under the first part of the title given above. It is devoted in part to the classical theories of the differential geometry of curves and surfaces, and in part to some very interesting special chapters of these theories with which the author himself has been especially occupied. In his preface Professor Blaschke says that this first volume contains a presentation of the properties of curves and surfaces which are invariant under the group of motions, that the second will be devoted to affine differential geometry, and that a third will present the geometrical theories of Riemann and Weyl, which are so closely related to the Einstein theory of gravitation.

There are two interesting features of the book that attract one's attention at the very start. The first is the consistent use of vector notations. All of us who have lectured on differential geometry have doubtless been impressed with the great economies in presentation which these notations would afford. I have myself hesitated to use them in lecture courses because of the loss of time necessitated at the beginning of a course by explanations to hearers who have had no experience with vector notations, and because of the slight element of awe and mystification which these notations seem to arouse in the minds of those who have had only a limited acquaintance with them. After reading Professor Blaschke's book, I have grave doubts of the correctness of my attitude. The properties of vectors which one needs in differential geometry are few and simple, and he has demonstrated that they may be clearly and concisely explained as the occasions for their use arise. When one considers the many applications of vector analysis in other domains as well as in geometry, it seems clear that we should acquaint our students with the elements of the subject at the earliest possible moment. Mathematical physicists usually shy away from abstract mathematical notations, as they did from the non-euclidean theories of space before the recent revolution. Is it not curious that they should be the leading advocates of the vector analysis notations which mathematicians in equally appropriate situations have in many cases studiously avoided?

The second outstanding feature of the book is the fifth chapter, devoted to the macroscopic properties of surfaces as contrasted to the differential properties with which most of us are more familiar. The only simply closed analytic surfaces with constant Gaussian curvature are spheres. If we designate an analytic simply closed convex surface by the name oval surface, then the only oval surfaces with constant mean curvatures are spheres. On every oval surface there are at least three closed geodesics and the image of each of them on the Gaussian sphere of the surface bisects the area of the sphere. These are samples of the theorems which this chapter contains. At the present time comparatively few results in the macroscopic theory of surfaces have been attained, most of them being properties of oval surfaces. The proofs which lead to them, as Professor Blaschke says in his preface, are among the most difficult known in any branch of analysis. It seems to me that the author has presented these proofs on the whole with great clearness, and we may feel confident that in this domain, in which he himself has had great interest, a comprehensive summary of known results has been given.

Throughout the book one notices frequent applications of the theory of the calculus of variations. This is inevitable in one form or another in the treatment of geodesics and minimal surfaces, but the author is well acquainted with the theory and apparently welcomes opportunities to apply this knowledge. In no case are theorems used, however, which are not established in the text. One may look forward to still further interesting applications of the calculus of variations in the third volume promised by the author.

The book is written in concise and "snappy" style, but the sequences of logical steps are clear and the text always interesting. References to original sources and historical remarks are frequent. The latter are sometimes more than mere paragraphs. On pages 80–81 and 123, for example, are very interesting historical sketches devoted to Monge and Gauss. In a number of places in the book computations are left to the reader. These are sometimes of considerable extent, but as I read the book I found only a few instances in which additional suggestions from the author would have helped to a more rapid understanding. At the ends of chapters are lists of problems and of theorems to be proved by the reader. These are in many cases not elementary, but references are usually given. I should regard the lists as on the average unsuited to the needs of beginning students, but suggestive of much interesting study and reading outside the limits of the book itself for readers of all types.

Chapter I is devoted to vector notations and the differential geometry of twisted curves, with applications to Bertrand curves, "Böschungslinien," evolutes and involutes, and isotropic curves. Böschungslinien are curves whose tangents make a constant angle with a fixed direction in space, and the author determines all such on a sphere or a paraboloid of revolution. In § 9 a foretaste of the later macroscopic chapter is given by a proof of the theorem that every oval plane curve has at least four vertices, that is, four points where the rate of change of the curvature with respect to the length of arc is zero. One misses, in this chapter, the theory of the moving trihedron whose axes lie in the tangent, principal normal, and binormal of a given twisted curve, and by means of which the properties of many curves associated point for point with the given one can be very simply deduced.

Chapter II is in many respects a calculus of variations chapter. The author begins by showing how the binormal and first curvature κ of a twisted curve can be defined by the first variation of the length integral along a curve, and considers the more general problem suggested by Radon of minimizing an integral of the form $I = \int \varphi(\kappa)ds$ where s is the length of arc. He then studies the isoperimetric properties of the circle, proving that the circle is actually the curve of given length enclosing a maximum area by the methods of Frobenius and Hurwitz. One wonders a little at this point why this material is introduced here. But it turns out that in Chapter VI the author gives two very interesting generalizations, for minimal surfaces and for the sphere, of the formula which is fundamental in Frobenius' proof, and the more general results of the very elegant proof of Hurwitz are also applied in a later section. The last three sections of the chapter are devoted to two theorems of Schwarz concerning minimum properties of twisted curves with constant curvature $\kappa = 1$.

In the seventy pages of Chapters III and IV the author presents more of the theory of surfaces than I should have thought possible in such a limited space. In the former of them one finds the theories of lines of curvature, asymptotic lines and conjugate systems, the Gaussian sphere, the formulas of Gauss and Codazzi, and related subjects. The latter is devoted to the elements of the theory of applicable surfaces, geodesics and related coordinate systems, Beltrami's differential invariants, isothermal parameters and conformal representations of one surface on another. Although the presentation is concise it seems to me unusually readable and interesting.

I have mentioned already the character of the contents of Chapter V, which is one of the most interesting in the book. There are in all some nine macroscopic theorems, six concerning oval surfaces, two concerning closed surfaces, and one concerning surfaces of constant curvature $K = -1$. In § 86 the author describes in some detail a problem which is still unsolved. On every geodesic line there corresponds in general to every point A a conjugate point A'. Every arc AB not containing A' is shorter than all other arcs near it joining A and B, while arcs AB containing A' between A and B do not have this property. On a sphere the geodesics are great circles and the conjugate point A' is diametrically opposite to A on the surface. The problem of § 86 is to determine the surfaces on which the lengths of the geodesic arcs AA' are all the same. It seems likely that these surfaces are necessarily spheres, but the author at present knows no complete proof of this surmise. He describes in very interesting fashion the results attained and the difficulties remaining to be overcome.

Chapter VI is entitled "Extreme bei Flächen" and contains an introduction to the theory of minimal surfaces and a study of the isoperimetric properties of the sphere. For minimal surfaces the equations of Weierstrass and Study are derived, besides Schwartz's formula expressing the area of a minimal surface as a line integral along its contour, and the necessary condition for a minimum which he deduced from the second variation of the area integral on the surface and which corresponds to Jacobi's condition in the calculus of variations. A discussion of the problem of the determination of a minimal surface by a curve and a strip of normals on the surface, with some allied theorems, is also given. Among all surfaces with given area the sphere is the one which encloses a maximum volume, but it is not a simple matter to prove this property. The author uses a process of symmetrization of oval surfaces due to Steiner in order to prove that the maximizing surface if it exists must be a sphere. But there still remains the necessity of proving that a minimizing surface actually exists. Existence proofs have been made by Schwartz (1884), Minkowski (1903), Blaschke (1916), and Gross (1917). The proof here given is the last one, which is a modification of that of Blaschke. In the course of this proof the relation $O^3 - 36\pi V^2 \geqq 0$ is deduced for the surface area O and the volume V of an oval surface, the equality holding only when the surface is a sphere. This is a generalization of a similar relation $L^2 - 4\pi F \geqq 0$ used in an earlier chapter in the Frobenius proof of the isoperimetric property of a circle, L being the length of a plane oval and F the area which it encloses. As a second generalization the author shows also that Frobenius' formula holds for a simply closed curve in space provided that L is the length of the curve and that F is interpreted as the surface area of the minimal surface passing through the curve.

The final chapter of the book is devoted to line geometry, especially the theory of ruled surfaces and congruences. Every straight line is uniquely determined by a unit vector $\boldsymbol{a}$ and a second vector $\boldsymbol{b}$ orthogonal to the first, by means of the equations $\boldsymbol{x} \times \boldsymbol{a} = \bar{\boldsymbol{a}}$, where $\boldsymbol{x} = (x_1, x_2, x_3)$ is a variable point on the line. By the equation $\mathfrak{A} = \boldsymbol{a} + \epsilon\bar{\boldsymbol{a}}$ the straight line furthermore determines a unique vector $\mathfrak{A}$ with components in the linear algebra of Clifford whose elements have the form $\boldsymbol{a} + \epsilon\boldsymbol{b}$, $\boldsymbol{a}$ and $\boldsymbol{b}$ being real numbers and ϵ a unit such that $\epsilon^2 = 0$. It is readily provable that every such vector $\mathfrak{A}$ is a unit vector, and conversely every unit vector $\mathfrak{A}^2 = (\boldsymbol{a} + \epsilon\boldsymbol{b})^2 = 1$ has its real vectors $\boldsymbol{a}$, $\bar{\boldsymbol{a}}$ satisfying $\boldsymbol{a}^2 = 1$, $\boldsymbol{a}\bar{\boldsymbol{a}} = 0$, and therefore determines uniquely a straight line in space. By this most interesting device of Study the lines of space are set into one-to-one correspondence with the points on a unit sphere in the Clifford space, the Clifford space being the totality of points $\mathfrak{A} = (\mathfrak{A}_1, \mathfrak{A}_2, \mathfrak{A}_3)$ whose coordinates $\mathfrak{A}_1, \mathfrak{A}_2, \mathfrak{A}_3$ are numbers in the Clifford algebra. A rotation in the Clifford space determines uniquely a motion in the original $\boldsymbol{x}$-space, and conversely, so that the determination of the invariants under rotations in the former is equivalent to the determination of invariants under the group of motions in the latter. A ruled surface is now defined by a vector $\mathfrak{A}(t)$ whose elements are functions of a single parameter t, and a congruence by a vector $\mathfrak{A}(u, v)$ depending upon two parameters. It is altogether fascinating to see how the usual properties of these surfaces and congruences fall out naturally and readily under this new aspect.

G. A. BLISS