BECK ON COORDINATE GEOMETRY

Koordinaten Geometrie. Volume 1, *Die Ebene.* By Hans Beck. Berlin, Julius Springer, 1919. x + 432 pp.

A celebrated New York divine of an earlier generation was once asked if he believed that he had a soul. He replied: "I do not understand the bearing of the question. I *am* a soul, I *have* a body."

Exactly the same distinction of verbs appears in the study of geometry. For historic or didactic reasons the student begins with synthetic geometry. The point is an undefined datum, about which more or less explicit assumptions are made. The geometry of points, lines, circles, etc., is carried to a considerable distance without any serious analytic help. After this analytic geometry is begun, and the student learns that, in order to present his points in well-bred society, he must clothe each with two numbers, called "coordinates." A moment's reflection shows him, however, that this is another case where "the apparel proclaims the man." What counts is not the undefined point, but the definite coordinates, the point is a sort of useless Kantian "thing-in-itself" that might as well be discarded.

The first writer to set this in perfectly clear light was Klein in his *Erlanger Program.* Here are various manageable groups of quantities, independent quantities, homogeneous quantities, quantities connected by certain relations. Here, further, are certain groups of transformations, which operate on these quantities; here are invariants and covariants of the groups. There are all sorts of conceivable sets of names to attach to the quantities, groups, invariants, etc. It is the path of wisdom to choose names which fit well into our geometric preconceptions and forms of speech. and to interpret in terms of them the results of our analytic manipulations, but we are at liberty at any moment to throw our vocabulary overboard and introduce new terms. Two geometries are equivalent if built on the same transformations and invariants, regardless of the shape of the letters used, or the names attached to them. At the very outset we do not say that a point *has* two coordinates, but that it *is* two coordinates. When the number field for the coordinates is real, this form of speech may seem a little artificial, but it is certainly legitimate. When, however, the number field is complex, a so-called "geometric" equivalent corresponding to the complex coordinates is bound to be more or less artificial, and the complete identification of point and coordinates by defining the former as nothing more nor less than the latter is not only natural but almost inevitable.

The present work, which takes this point of view from the outset, epitomizes and completes the results reached by a number of geometers during the last half century, the greatest of whom is Study. The fact is that the discerning reader will continually perceive the figure of Study behind a large part of what Beck has written to the extent that he may even feel inclined to exclaim:

"The voice is Jacob's voice, but the hands are the hands of Esau."

This means, of course, that the detail is very carefully done; let us look at it somewhat more closely.

In Chapter I a point is defined as a pair of complex coordinates, a line as the totality of points whose coordinates satisfy a linear equation. A sharp distinction is drawn between an isotropic and a non-isotropic line; through each point will pass just two lines of the former category. If a non-isotropic line l be given, we obtain an involutory collineation by exchanging each point of the plane with that other point whose two isotropics meet l in the same two places. This is called a *reflection* in the line l. The distance of two points is defined, analytically of course, and it is shown that distances are invariant under such reflections. Translations are obtained as the products of reflections in pairs of parallel lines, rotations as the products of reflections in intersecting lines; the whole euclidean group for the plane is built in a simple and satisfactory manner.

In Chapter II the center of interest is the straight line. The line has three homogeneous coordinates, and the idea of homogeneity is thoroughly discussed, although, curiously enough, homogeneous point-coordinates are not introduced until thirty pages later. The tangent of the angle of two lines is defined by the usual rational formula, but the tangent of the angle is not the function we want, since its period is π; we want a function with the period 2π, namely, the cosine; and this, unfortunately, is not rationally expressible in terms of the coordinates of the line. To avoid the difficulty, resort is had to the familiar process of orienting the line. Every set of numbers, not all zero, proportional to four quantities $u_0 : u_1 : u_2 : u_3$, where $-u_0^2 + u_1^2 + u_2^2 + u_3^2 = 0$, shall be called an *oriented line* or a *spear* (German, *Speer*). This spear is said to be *on* the line $u_1x + u_2y + u_3 = 0$. The cosine of the angle of two spears is given by the formula

$$\cos (u, v) = \frac{u_1v_1 + u_2v_2}{u_0v_0}.$$

The distinction between a line and a spear is most carefully made, and it is pointed out how positive and negative distances may be rationally separated on a spear. The latter part of the chapter deals with vectors and *staffs*, i.e., vectors limited to a single line; it is a real pity that we have no good English name for these quantities except, perhaps, forces.

Chapter III is headed *Collineations*, and deals with the fundamental properties of these transformations, and with their classification. I must confess that at this point the treatment seems to me somewhat labored. It is a point of honor with the author to put every formula not only in absolutely accurate form, but in its most general form. He scorns such artifices as simplifying a problem by a change of coordinates. And when you come to think of it, he may not do this; for if you do not say that a point *has* two coordinates, but that it *is* two coordinates, why then you cannot change the coordinates and have the same point. He has such analytical skill that he suffers less from this puritanic point of view than would most writers, but he devotes fourteen very full pages to exhibiting an example of each of five general types of plane collineations, and then four more pages to the characteristic equation and the proof that all possible types have been covered. The chapter ends with sections discussing affine transformations, stretchings, and systems of circles, oriented and non-oriented.

The fourth chapter is headed *Correlations*, but this is a misnomer. The principal use made of correlations is to define a conic according to Steiner's scheme. The bulk of the chapter is given to those collineations, happily called *automorphic*, which leave a conic in place. These are turned over on every side, and expressed in various ways, including quaternion form. What the writer is preparing for is the Cayleyan projective-metric, and this is taken up with great care. The radicals in both distance and angle formulas are avoided by the startling device of orienting both points and lines. The non-euclidean trigonometry is developed very prettily, and the already familiar euclidean metrical formulas are reached by a limiting process. A curious use of words is that whereby Lobachevski parallels, lines intersecting on the absolute conic, are defined as *paratactic;* this term was first introduced by Study in 1902 * to denote Clifford parallels, skew lines at a constant distance in three-dimensional space.

The domain in the first four chapters is the complex one. Chapter V, the last, concerns itself with real figures, real points, real collineations, real conics, etc. Circle geometry in the Gauss plane appears near the close.

And what shall we say of the book as a whole? The sketchy account above can give but the faintest idea of the amount of material; for, besides the main text, there are endless pages of problems and suggestions in small type. As mentioned above, every one of the bewildering assortment of formulas is stated with minute care, developed with conscientious rigor, and exhibited in its most general form. This is no easy accomplishment, as every geometer knows. One has the impression of a compendium, where the last word is said in deducing a large number of formulas; or, to change the metaphor, a vast arsenal where many perfect tools are built and stored for the benefit of future attacks on the difficulties of geometry. The writer could not accomplish his result without paying the price. He paid it, and the reader must pay also. The book is never obscure, the manipulations are not over-difficult, but the total effect is rather overwhelming. There are countless formulas developed for no apparent reason; there are scarcely any theorems or applications. The method is always the same: careful, straightforward conscientious algebraic analysis, with skillful use of a small number of simple algebraic identities, much as in the classical invariant theory. It is all very perfect, also very fatiguing. One has the impression that the writer is somewhat lacking in suppleness and imagination. A reviewer once said of one of Salmon's books that the reader had the instinctive feeling that the writer always hit on just the right method to solve each problem. The late Gaston Darboux once told me that it seemed to him that the essence of geometry consisted in finding in each individual problem the best method for its solution. But Beck will none of all this. Rigorous formalism must be carried through to the end, though the heavens fall.

J. L. COOLIDGE

* JAHRESBERICHT DER VEREINIGUNG, vol. 11.