bitrarily small, then a set of n rational fractions $x_1/y, \cdots, x_n/y$ can be found such that $|\alpha_i - x_i/y| < \epsilon$ and also $< \delta/y^{1+1/n}$ $(i = 1, 2, \cdots, n)$, where δ is a fixed number depending upon n only (cf. TRANSACTIONS OF THIS SOCIETY, vol. 15 (1914), pp. 234–5; L. E. Dickson, *History of the Theory of Numbers*, vol. 2, p. 95 ff.). A. Hurwitz has proved (loc. cit.) that $\delta = 1/\sqrt{5}$ for $n = 2$; and H. F. Blichfeldt that $\delta \geqq 1/\sqrt[3]{23}$ for $n = 3$ (proof not yet published). In MATHEMATISCHE ANNALEN, vol. 83 (1921), p. 77 ff., there is a proof by O. Perron that $\delta > (1/n)[0.35/(n + 1)]^n$ approximately. The present writer, using Perron's method with refinements based on the principles of the geometry of numbers, obtains the somewhat better result $\delta > (N/\sqrt{n})(2/\sqrt{n})^n$ approximately, where N is a fixed number.

B. A. BERNSTEIN,
Secretary of the Section.

THE EASTER MEETING OF THE SOCIETY

The two hundred twenty-second regular meeting of the American Mathematical Society, being the seventeenth regular Western meeting, and the forty-ninth regular meeting of the Chicago Section, was held at the University of Chicago on Friday and Saturday, April 14 and 15, 1922, in honor of the twenty-fifth anniversary of the Chicago Section. The attendance at these meetings was approximately one hundred fifty, among whom were the following one hundred four members of the Society:

E. S. Allen, F. E. Allen, Baker, Beckwith, Bliss, Blumberg, Bradshaw, Brahana, Brooke, Bussey, C. C. Camp, Carmichael, Chapman, Chittenden, Coble, Crathorne, H. B. Curtis, Curtiss, Dalaker, H. T. Davis, Denton, Dickson, Doll, Dowling, Dresden, Escott, Everett, Eversull, Feldstein-Tartakovsky, Feltges, Fields, Fry, Gibbens, Glenn, Gouwens, Green, W. L. Hart, M. G. Haseman, E. R. Hedrick, Hildebrandt, Hoar, Hodge, Ingraham, Dunham Jackson, Kinney, Lane, Lennes, Logsdon, Lunn, McGaw, M. M. McKelvey, J. V. McKelvey, N. B. MacLean, MacMillan, March, Marshall, T. E. Mason, Meacham, B. I. Miller, E. B. Miller, G. A. Miller, Miser, C. N. Moore, E. H. Moore, E. J. Moulton, F. R. Moulton, A. L. Nelson, Newson, Olson, C. I. Palmer, Pitcher, J. F. Reilly, R. G. D. Richardson, Rider, H. L. Rietz, Risley, Roever, Roman, Schottenfels, Schweitzer, Shaw, W. G. Simon, Skinner, Slaught, Edwin R. Smith, Stecker, Steimley, Stouffer, E. L. Thompson, B. M. Turner, J. S. Turner,

Underhill, Wahlin, Walton, Warren Weaver, Wilczynski, Wilder, K. P. Williams, F. E. Wood, Roscoe Woods, J. W. Young, J. W. A. Young, Zehring, Ziwet.

The Council announced the election of the following persons to membership in the Society:

Professor Horace Seely Brown, Hamilton College;
Miss Bess Marie Eversull, University of Cincinnati;
Rev. Francis Joseph Gerst, Loyola College, Baltimore;
Professor Eleanora Harris, Central Missouri State Teachers College;
Professor Albert Anton Heinz, Tsing Hua College;
Miss Anna Margaret Mullikin, Oak Lane, Pa.;
Mr. Percy Charles Herbert Papps, Mutual Benefit Life Insurance Company, Newark;
Professor Harry Munson Showman, University of California, Southern Branch;
Mr. Willis Whited, Consulting Bridge Engineer, Pennsylvania State Highway Department;
Professor Emma Kirtland Whiton, University of Redlands.

Professor A. B. Coble was reelected a member of the Editorial Committee of the TRANSACTIONS, for a term of three years beginning October 1, 1922. The President was requested to appoint a committee to prepare nominations for officers and other members of the Council to be elected at the annual meeting in December.

On behalf of the committee that had collected a fund in honor of Professor E. H. Moore, Professor Arnold Dresden tendered the fund to the Society. The trust was accepted, under the conditions of the bequest. A complete account is given elsewhere in the present number of this BULLETIN.

The sessions of Friday morning and Saturday were devoted to the reading of the papers listed below. On Friday afternoon Professor A. B. Coble gave the symposium lecture on *Cremona transformations and applications to algebra, geometry, and modular functions*. The program for Saturday morning was made up of three papers delivered by invitation in special celebration of the anniversary of the Chicago Section.

On Friday evening a dinner was held at the Quadrangle Club at which one hundred six persons were present. Toasts were responded to by Professor Coble who spoke for

the Chicago Section, Professor Richardson who spoke for the Society, Professor Ziwet and Mrs. H. B. Newson who spoke for the early members of the Chicago Section, and Professor Hildebrandt who spoke on Professor E. H. Moore as mathematician, teacher and man. At the close of Professor Hildebrandt's remarks, Professor Dresden presented to Professor Moore a beautifully illuminated and bound manuscript containing a statement of the establishment of the Eliakim Hastings Moore fund, of which mention has already been made.

At the meetings of the Society, President G. A. Bliss presided, relieved by Vice-President R. D. Carmichael and Professors A. B. Coble, L. E. Dickson, and E. H. Moore.

Titles and abstracts of the papers read at the various sessions follow below. Miss Carlson's paper was read by Professor Jackson. The papers of Dr. Zeldin, Dr. Wiener, Professors Copeland, Glenn, Crum, and R. L. Moore, Dr. Woods, Dr. Speiser, Mr. Wilder, and Professor Hart were read by title. Professor Carr was introduced by Professor MacMillan, Dr. Speiser by Professor Dickson, and Mr. Wilder by Professor R. L. Moore.

1. Professor R. D. Carmichael: *Abstract definitions of the symmetric and alternating groups and certain other permutation groups.*

The most interesting of the theorems given in this paper is the following: If k operations are subject to the sole defining conditions that each is of order three and the product of each pair of them is of order two, then they generate a group which is simply isomorphic with the alternating group of degree $k + 2$. About a score of theorems are given, each of which affords the abstract definition of some infinite class of permutation groups, several of these theorems being devoted to various abstract definitions of the symmetric and alternating groups.

2. Professor D. R. Curtiss: *On the zeros of successive polars of a binary form.*

A theorem due to J. H. Grace states that if a binary form written as $f(z)$ in non-homogeneous notation is apolar to a

form $\phi(z)$, then it has a zero within any circle enclosing all the roots of $\phi(z)$. In the present paper it is shown that this is equivalent to the following statement, which has been proved in another way by Dr. J. L. Walsh: Let $z_1, z_2, \cdots, z_n$ lie within a circular region C and satisfy an equation

$$F(z_1, z_2, \cdots, z_n) = 0,$$

where F is symmetric in the z's and is a linear function of each z_i; then the equation $F(z, z, \cdots, z) = 0$ has at least one root in C. A proof is given by mathematical induction, and the theorem is shown to be a special case of one regarding the zeros of successive polars of a form which can be deduced from results due to Laguerre.

3. Professor D. R. Curtiss: *Relations between kindred P and Q functions.*

In his lectures on the hypergeometric function, Klein stated without proof the theorem that every Q function of order k could be expressed linearly with constant coefficients in terms of $k + 1$ kindred P functions. This statement would seem to have been based on a count of constants, and is, in fact, untrue in certain special cases. The present author gives a proof of its correctness, with a method for obtaining such representations, in case the families concerned are irreducible. This method also applies to many reducible cases. As a corollary it follows that in these reducible cases, as well as in the irreducible case, every $k + 2$ Q functions of order k which have the same exponents are linearly dependent. A similar examination is made of the relations between a Q function and a kindred P function and its derivative.

4. Professor C. N. Moore: *On the equivalence of the Cesàro and Hölder means for multiple limits.*

In a paper previously presented to the Society (this BULLETIN, vol. 25 (1919), p. 257) the author has established a theorem with regard to generalized limits in General Analysis which includes as special cases the Knopp-Schnee-Ford theorem with regard to the equivalence of the Cesàro and Hölder means for series, the analogous theorem of Landau for integrals, and other similar theorems dealing with simple limits. In the present paper this theorem is extended to the case of multiple limits, so that it now includes as special cases corresponding theorems relating to multiple series, multiple integrals, and other multiple limits. The main

part of the proof consists in the establishment of an identical relation between Cesàro and Hölder means for multiple limits, which, though considerably more complicated for the general case, is analogous to Schur's identity (MATHEMATISCHE ANNALEN, vol. 74 (1913), p. 452) for the case of simple series.

5. Miss Bess M. Eversull: *On convergence factors in triple series and the triple Fourier series.*

In this paper it is shown that if convergence factors satisfying certain conditions are introduced into a triple series summable ($C1$), the resulting series will converge and will approach the value to which the original series is summable, as the convergence factors approach unity. It is also shown that the Fourier development of a function of three variables that is finite and integrable (Lebesgue) will be summable ($C1$) to the value of the function at every interior point of a region throughout which the function is continuous. The foregoing results are used to show that the formal developments that arise in discussing certain problems in the flow of heat really furnish the desired solution of the physical problem.

6. Professor E. B. Stouffer: *Independent sets of coaxial minors of determinants.*

It is known that there cannot be more than $n^2 - n + 1$ independent coaxial minors of a determinant of the nth order. The author determines two such sets of coaxial minors, and obtains the expressions for the elements of the determinant in terms of the minors belonging to each independent set. If one or more of the columns of the original determinant are replaced by corresponding columns from other determinants of the nth order with independent elements, a set of determinants is obtained for which a complete system of independent coaxial minors is determined. The independence of certain sets of sums of coaxial minors of determinants is also proved. This is of importance in invariant theory.

7. Professor P. R. Rider: *On the minimizing of a class of definite integrals.*

Several problems in the calculus of variations lead to the consideration of a definite integral of the form $\int_{x_0}^{x_1} [(1+y'^2)^m/y'']dx$. These are of peculiar interest because the second derivative appears in the integrand, and comparatively few problems of that kind have been completely solved. For example,

Euler's historic problem of finding the curve which with its evolute encloses a minimum area gives rise to the particular case $m = 2$ in the above integral. The case $m = 1$ arises in the problem, treated by Rider and Dunkel, of obtaining the curve which with its caustic encloses a minimum area. In finding the curves of minimum mean radius of curvature with respect to the arc and with respect to the abscissa, we are led to the cases $m = 2$ and $m = 3/2$ respectively. In this paper the author obtains the equations of the extremals or minimizing curves for these cases, and proves sufficient conditions for a minimum. The question of the determination of the arbitrary constants that occur in the equations of the extremals is taken up, and it is shown that an extremal can always be passed through two given points and be made to have prescribed slopes at these points. Various cases of variable end conditions are considered. The theory is illustrated by means of the curve of minimum mean radius of curvature with respect to x.

8. Miss Elizabeth Carlson: *On the approximate representation of periodic functions of two variables.*

It has been proved by D. Jackson (TRANSACTIONS OF THIS SOCIETY, vol. 22) that if $f(x)$ is a given function of period 2π, if n is a given positive integer, and if m is a real number greater than 1, then there exists one and only one trigonometric sum $T_{mn}(x)$ of order n, for which the integral of the mth power of the absolute value of the difference between $f(x)$ and $T_{mn}(x)$ reaches a minimum. The corresponding theorems for functions of two variables are proved in this paper. Also, a sufficient condition for the convergence of $T_{mn}(x,y)$ to $f(x,y)$ is found to be that $\lim_{\delta \to 0} \omega(\delta)/\delta^{2/m} = 0$, where $\omega(\delta)$ is the modulus of continuity of $f(x,y)$.

9. Professor G. A. Miller: *Substitution groups whose cycles of the same order contain a given number of letters.*

The total number of letters in all the cycles of order k contained in the group G may be found as follows: Select a set of cycles of order k composed of all the different cyles of this order found in G. Let λ be the number of the complete sets of conjugates under G contained in this set. The total number of letters in all the cycles of order k found in G is then λg, g being the order of G. The triply transitive group of degree 6 and order 120 is the only group which is not

symmetric but has the property that the total number of letters in all of its cycles of the same order is exactly equal to the order of the group for every cycle which appears in the group. A necessary and sufficient condition that a transitive group of degree n contain a complete set of distinct conjugate cycles whose total number of letters is less than the order of the group is that the class of this transitive group be less than $n - 1$. There are exactly six transitive groups which are not alternating but have the property that the total number of letters in all the cycles of the same order save one is the order of the group, while for this one the total number of letters is twice the order of the group.

10. Dr. S. D. Zeldin: *Conformal transformations of linear homogeneous difference equations and their invariants.*

In this paper Dr. Zeldin shows that the most general conformal transformation which leaves invariant a linear homogeneous difference equation of the type $p_0(v)u_{v+n} + p_1(v)u_{v+n-1} + \cdots + p_n(v)u_v = 0$, where u and v are the circular coordinates of a point in two dimensional space, is of the form $v = \varphi(\xi)$, $u_v = cu_\xi$, where $\varphi(\xi)$ satisfies one of the two equations $\varphi(\xi + 1) - \varphi(\xi) = \pm 1$, and c is an arbitrary constant. He also determines the functions of the p's which form independent absolute invariants.

11. Dr. Norbert Wiener: *A new form of integral expansion.*

The author develops the formula

$$f(x) = \lim_{n\to\infty} \frac{1}{n!\int_{-1}^{1} e^{1/(x^2-1)}dx} \int_0^1 \frac{d^{n+1}}{du^{n+1}} \{[(u - x)^n + (u + x)^n]\int_{-1}^{u} e^{1/(u^2-1)}du\} f(u)du$$

for even functions $f(x)$ which are uniformly analytic over $(-1, 1)$. He discusses the conditions of validity of this formula, contrasting it with Fourier expansions. He shows that if it is valid for $x = 0$, $f(x)$ is uniquely determined by its values in the neighborhood of $x = 1$.

12. Professor Lennie P. Copeland: *Note on certain semi-invariants of n-lines.*

In this paper the semi-invariantive conditions that one n-line be a pole-polygon of a second n-line are determined. It is proved that there exist $3 + (m - 2)(m - 1)$ necessary and sufficient conditions that two n-lines be mutual pole-polygons. If one n-line is a pencil, the second is also, and they have coincident vertices.

13. Professor O. E. Glenn: *Residues of figurate numbers.*

Systems of residues (modulo p, a prime) may be developed as follows: Select a sequence (r) of integers, infinite one way, satisfying any law, and employ its terms as the initial numbers of appropriate orders of differences in a table of differences. The numbers in this table are of the form

$$N = \sum_j r_j \binom{m}{j}$$

and their residues are arranged, in general, in symmetrically related geometric figures that may be delimited by formulas. Thus we obtain, for instance, all numbers of the type of N which are divisible by p. Among the results of the paper is given a determination of all combinatory numbers

$$\binom{m}{j}$$

which are divisible by p^w and not by p^{w+1}.

14. Professor W. L. Crum: *Inter-variate correlation and the successive measures of dispersion in an ordered statistical series.*

In this paper formulas are developed for calculating the coefficients of correlation between the variates and (1) those next preceding, (2) those preceding by two time-intervals, and so on up to k time-intervals. It is shown that these formulas are expressible in terms of the standard deviations of the series of original items and of their successive finite differences. Several of the correlation coefficients are worked out for particular illustrations from historical statistics. The writer then presents the hypothesis that we may measure dispersion in an ordered series by a succession of numbers, beginning with the standard deviation and including the successive inter-variate correlation coefficients as far as we like. If the correlation coefficients fall off rapidly, the series is highly unstable; if they remain nearly equal to the basic coefficient, the series is relatively stable.

15. Professor W. L. Crum: *Inter-variate partial regression equations in an ordered statistical series.*

This paper seeks to examine the phenomena of fluctuation in an ordered series from a somewhat different point of view from the usual attempt to discover fundamental periodicities. Instead of considering that there is a more or less closely

periodic tendency for values of the variable to repeat themselves, the paper studies the hypothesis that there is an average tendency for the items of a series to bear a definite relation to the items next preceding, to those next-but-one preceding, etc. The obvious method for discovering such a relation is by the use of partial regression equations, and general properties of these are developed for the kind of series under consideration. Application of the theoretical results is made to certain historical economic series with a view to testing the validity of the hypothesis and to determining its practical utility in the problem of prediction in time series.

16. Professor R. L. Moore: *Concerning relatively uniform convergence.*

In this paper the author shows that if a sequence of measurable functions converges at every point of an interval I, then it converges relatively uniformly over some subset of I whose measure is identical with that of I.

17. Professor R. L. Moore: *On the cut-points of continuous curves and of other connected point sets in space of two dimensions.*

A proper subset K of a point set M is said to disconnect M in the strong sense provided $M - K$ is not connected. It is said to disconnect M in the weak sense provided not every two points of $M - K$ can be joined by a *closed* and connected subset of $M - K$. The present paper contains, among other results, the following:

(I) No bounded, closed and connected point set M contains a closed and connected subset K which contains an uncountable set of points each of which disconnects M, but not K, in the strong sense.

(II) In order that a closed, connected and bounded point set M shall be a continuous curve which contains no simple closed curve it is necessary and sufficient that every closed and connected subset of M contain uncountably many points each of which disconnects M (in the strong sense).

(III) In order that the continuous curve M shall contain no simple closed curve it is necessary and sufficient that if K denotes the set of all those points $[X]$ of M such that X disconnects M then no subset of K shall disconnect M, even in the weak sense.

18. Professor F. E. Carr: *A solution of a spinning oblate spheroid two body problem.*

The problem deals with the precessional movement of the spheroid and was suggested by the non-rigorous treatment which the earth's precession received from Poisson, Serret and others. The solution for p, the precession, has the form $p = P_0 t + P$, where P_0 is a power series with constant coefficients and P a power series with periodic coefficients, in a parameter denoting the eccentricity of the spheroid. The solution for the motion of the second body is analogous to and a generalization of one of the solutions obtained by W. D. MacMillan, and reduces to that solution for a special case. The solution is not general, but arises from a special set of initial conditions. When applied to the system of the earth and moon and the earth and sun, the solution gives results that agree closely with observed values.

19. Dr. Roscoe Woods: *The elliptic modular functions associated with the elliptic norm curve E^7.*

Professor Bessie I. Miller has discussed (TRANSACTIONS OF THIS SOCIETY, vol. 17 (1916), p. 259) the elliptic norm curves E^n for $n = 3$, 4, and 5, for which cases the genus of the associated modular group is zero. In the present paper, which will appear in full in an early number of the TRANSACTIONS OF THIS SOCIETY, the case $n = 7$, for which the genus is 3 and which is fairly typical of the general case, is considered.

20. Dr. Andreas Speiser: *Die Zerlegung von Primzahlen in algebraischen Zahlkörpern.*

In this paper, which will appear in an early number of the TRANSACTIONS OF THIS SOCIETY, the problem of the decomposition of a prime number in an algebraic number domain is shown to be identical with a certain problem in the theory of linear substitutions in a Galois field.

21. Professor G. A. Bliss: *A boundary value problem in the calculus of variations.*

For problems of the calculus of variations with fixed end-points it has been shown by several authors that the Jacobi necessary condition for a minimum is closely associated with a boundary value problem of the linear differential equation theory. In the present paper an analogous situation is studied for a case when the end-points are variable, and the correspondences between the two theories are exhibited.

22. Professor J. F. Reilly: *Certain generalizations of osculatory interpolation.*

The Sprague-King formula for osculatory interpolation makes use of differences to order five, and requires that the curve have second order contact with the partial interpolation curves at the extremities of the interval to which the formula is applicable. The author shows how this formula can be generalized first to make use of differences to order $2h + 1$, where h is a positive integer; and second to require that the curve have contact of order k with the partial interpolation curves. Further, it is indicated how these generalizations apply to formulas employing central differences of four different types: (1) the ordinary central differences of y_h; (2) the central differences relative to the interval between y_h and y_{h+1}; (3) the odd central differences relative to the interval between y_h and y_{h+1}, and the even central differences of y_h, as used by Gauss and Karup; and (4) the even central differences of y_h and y_{h+1}, as used by Everett and Buchanan.

23. Professor Arnold Dresden: *A report on the scientific work of the Chicago Section,* 1897–1922.

This paper appears in the present number of this BULLETIN.

24. Professor J. B. Shaw: *On functional transformations.*

The first part of the paper is concerned with the structure of linear operators of the type representable by integral transformations, the second part with functions of these operators, and the third part with algebras of such operators built upon a given fundamental complex.

Examples are used to introduce the notions such as: (1) $\int_{-1}^{+1}(9x^2y^2 - 3x^2 - 3y^2 + 2xy + 2)()dy$ with fundamental functions 1, x, $(3x^2 - 1)$. (2) $\int_{-1}^{+1}(15x^2y^3 - 9x^2y + 3xy^2 - 5y^3 + 4y - x)()dy$ with fundamental functions 1, x, $(3x^2 - 1)$, $(5y^3 - 3y)$. In case (1) the transformation converts the functions into numerical multiples of themselves; in case (2) it converts each into a multiple of the preceding, and 1 into 0. (3) $\int_0^1[x(1 - y), y(1 - x)]()dy$ with normalized fundamental functions $\sqrt{2}\sin n\pi y/n^2\pi^2$, a denumerable infinity. (4) $\int_0^\infty \tfrac{1}{2}e^{-|x-y|}()dy$ with normalized fundamental functions $\sqrt{2}\cos(x\cot\alpha - \alpha)(\sqrt{\pi}\sin\alpha)$, a continuous infinity of fundamental functions. (5) $\int_0^x$ and $()dy$ with the

fundamental functions $x^{\alpha+n}$ where $0 \leqq \alpha < 1$, and n runs from 1 to ∞, integrally.

For each of these operators an identity transformation exists which gives the so called expansion of the function in terms of the fundamental functions.

In the second part of the paper the real character of the various solutions of integral equations is brought out, such as the Neumann solution, Fredholm's solution, Volterra's solution, and the linear algebra solution. In the third part the use of a given set of fundamental functions belonging to a given transformation in determining other related transformations is shown.

25. Professor E. H. Moore: *On the determinant of an hermitian matrix with quaternionic elements. Definition and elementary properties with applications.*

This paper will appear in an early number of this BULLETIN.

26. Dr. T. C. Fry: *Trigonometric expansions of aperiodic functions.*

At the summer meeting of the American Mathematical Society at Wellesley in September, 1921, a discussion arose regarding methods for determining the amplitudes of sinusoidal components in functions which are not themselves periodic. In the course of this discussion a paper by W. L. Hart (*On trigonometric series,* ANNALS OF MATHEMATICS (2), vol. 18, p. 99) was frequently mentioned and the criticism was made that the restrictions imposed by Hart rather seriously interfered with the usefulness of the results. In the present paper the author attacks this problem from the standpoint of the theory of divergent limits and removes the most serious of these restrictions.

27. Mr. R. S. Hoar: *Mathematical paradoxes involved in the new Bucyrus gasoline shovel.*

The new gasoline shovel just put out by the Bucyrus Company is the first one-engine shovel of any size. Its peculiar construction involves enough apparent mathematical paradoxes to furnish the material for a large part of a course in mechanics. For example, the digging part of the shovel is forced against the bank by the weight of the bank, rather than by the engine. Also, it has a one-part hoist which is

apparently a two-part hoist, and the direction of the hoisting force is at an angle of about 45 degrees from its apparent direction. The object of this paper is to place these paradoxes at the disposal of teachers of mechanics, and to inject a bit of mathematical recreation into the programme.

28. Professor E. W. Chittenden: *On permutable quadratic forms in infinitely many variables.*

The first part of the paper contains a determination of the properties of the most general limited quadratic form $K(x)$ in infinitely many variables permutable with a given limited form $H(x)$. The results obtained are applied in the second part of the paper to obtain the solutions of non-linear matricial equations.

29. Professor J. S. Turner: *A fundamental system of invariants of a modular group of transformations.*

In this paper the author considers the group H of all linear homogeneous transformations $x' \equiv ax + by$, $y' \equiv cx + dy$, $ad - bc \equiv 1 \pmod{p^2}$, where a, b, c, d are integers and p is an integral prime, and shows that any invariant $I(x, y)$ of this group is an invariant of the group G: $x' \equiv a_1x + b_1y$, $y' \equiv c_1x + d_1y$, $a_1d_1 - b_1c_1 \equiv 1 \pmod{p}$, where a_1, b_1, c_1, d_1 are integers. It is proved $I(x, y) \equiv R(L, Q) + pF(x, y) \pmod{p^2}$, where L, Q are the known fundamental invariants of the group G, and R, F denote rational and integral functions of their arguments with integral coefficients. By means of the transformation $x' \equiv x + py$, $y' \equiv y \pmod{p^2}$, it is then proved that $R(L, Q) \equiv R_1(L^p, Q^p)$, $F(x, y) \equiv F_1(L, Q) \pmod{p}$. Finally, it is proved that L^p, Q^p, $pL^\alpha Q^\beta$, where $\alpha, \beta = 0, 1, \cdots, p - 1$, α, β, not both zero, form a fundamental system of (independent) invariants of the group H.

30. Professor F. H. Hodge: *Note on a generalization of the strophoid.*

A circle tangent to the Y axis is drawn. This circle plays the same rôle in the determination of the curve in question as that taken by the Y axis in the ordinary construction of the strophoid. This leads to a curve of the sixth order. If the radius of the circle is increased indefinitely, this curve approaches the strophoid as the limiting case.

31. Professor E. P. Lane: *Ruled surfaces of Green-reciprocal congruences.*

The author considers an arbitrary non-developable surface S and associates with this surface an arbitrary Γ'-congruence and the reciprocal Γ-congruence, in the sense of G. M. Green. Corresponding to an arbitrary curve on S there is a ruled surface R' of Γ'; the lines of Γ reciprocal to the generators of R' form the corresponding ruled surface R. The present paper investigates the properties of such corresponding ruled surfaces. The formulas herein developed furnish, in particular, a very brief and elegant method of determining those ruled surfaces R' and R which are developable. The theorem of Green, that the conjugate of a Γ-tangent intersects the corresponding generator of the corresponding developable R in a focal point, is generalized in the form: *The conjugate of a tangent to the curve on S that corresponds to an arbitrary ruled surface R of the Γ-congruence intersects the corresponding generator of R at the point where R is touched by the corresponding tangent plane of S.* A class of curves called *intersector curves* is defined for R' and R. Applications are made to asymptotic ruled surfaces R' and R, which correspond to asymptotic curves on S. These, with the canonical quadric, serve to characterize the pair of covariant reciprocal congruences defined, in Green's notation, by the conditions $\alpha_u = \beta_v = 0$.

32. Professor K. P. Williams: *The Laplace-Poisson mixed equation.*

The author investigates the analytic character of the formal solutions which Borden has obtained for the Laplace-Poisson mixed equation

$$f'(x + 1) + p(x)f'(x) + q(x)f(x + 1) + r(x)f(x) = 0.$$

The asymptotic form of the solutions is also considered.

33. Mr. H. T. Davis: *A criterion from integral equations relating to the existence of solutions for the one-dimensional boundary value problem.*

In this paper the known existence of solutions of a linear homogeneous differential equation with a parameter of the Sturmian type for one set of boundary conditions is used to prove the existence of solutions which satisfy a second set of boundary conditions. If $G_1(x, t)$ is the Green's function for the first system, and $G_2(x, t)$ the Green's function for the

second system, then $\int_a^b [G_1(x, x) - G_2(x, x)]dx \neq 0$ is shown to be a sufficient condition that the values of the parameter for which solutions exist for the two problems shall either coincide or alternate with one another.

34. Mr. H. T. Davis: *A general criterion relating to the existence of solutions for the one-dimensional boundary value problem.*

Suppose F_1 is the determinant whose zeros are characteristic values of one boundary value problem, and F_2 the determinant belonging to a second boundary value problem. The product $F = F_1 \cdot F_2$ is used to furnish a criterion for the existence of solutions for one system, supposing the existence of solutions has already been proved for the other. An application is then made to the general self-adjoint system of second order.

35. Mr. R. L. Wilder: *A continuous curve in the rôle of a space.*

The author shows that if the set of points constituted by a continuous curve is regarded as a space S, then if certain terms are suitably defined, many theorems which are true in ordinary space hold true for the space S.

In particular, it is shown that Theorems 1–16 of R. L. Moore's paper *On the foundations of plane analysis situs* (TRANSACTIONS OF THIS SOCIETY, vol. 17 (1916), pp. 131–164) hold true. In addition, the following theorems are proved: (1) If R is a domain with respect to S, and B, the boundary of R with respect to S, is connected im kleinen, then any point P in B can be joined to any point in R by a simple continuous arc lying wholly in R except for the point P. (2) In order that a closed and bounded point set shall be connected im kleinen, it is necessary and sufficient that it be the sum of a finite (or vacuous) set of continuous curves, together with a finite (or vacuous) set of isolated points.

36. Professor N. J. Lennes: *Continuous transformations in analysis situs.*

For a given Jordan curve j and a fixed point P within it, this paper describes the construction of a set $[j]$ of Jordan curves such that (1) every curve lies within j and contains P as an interior point; (2) through every point within j, except P, there is one and only one curve of the set $[j]$. If any curve j_1 of the set $[j]$ is removed, the remainder of the set consists

of two sets of curves $[j]_1$ and $[j]_2$ such that (*a*) no point in either set is a limit point of points in the other; (*b*) every point of j_1 is a limit point of points in $[j]_1$ and also of points in $[j]_2$. The set of curves $[j]$ is said to constitute a *continuous transformation* of any one of its curves into any other or into the point P.

Such a set can be constructed transforming any plane curve into any other provided the curves have no point in common and one of the curves does not have a point interior to and another point exterior to the other curve.

The problem is proposed to construct a continuous transformation of a *doubly closed continuous curve not lying in a plane* into a plane Jordan curve. This problem, connected with the theory of "knots," is not completely solved.

37. Professor N. J. Lennes: *On the foundations of the theory of sets.*

In the MATHEMATISCHE ANNALEN, vol. 65, E. Zermelo gives a set of axioms for sets (Mengen), one of his purposes being to formulate a theory which should not involve the contradiction that has been shown to follow from Cantor's classical definition of "Menge." The purposes of the present paper are: (1) to show that though it follows from Zermelo's axioms that there are sets having cardinal numbers such as that of the continuum and higher, it cannot be shown that an arbitrary collection of objects more than finite in number is a set; (2) to modify Zermelo's theory so as to identify as a set any collection of objects having the same cardinal number as that of some Zermelo set; (3) to study the independence of Zermelo's axioms. It is proved that except for the "axiom of choice" (Axiom der Auswahl) Zermelo's axioms are independent. Similar independence proofs are given for the axioms with which it is proposed to replace Zermelo's set.

38. Professor L. E. Dickson: *An error in the theory of differential equations by Lie's method.*

In Lie-Scheffer's *Differentialgleichungen*, pp. 412–433, it is proved that every differential equation of the second order in two variables which is invariant under two infinitesimal transformations generating a two-parameter group can be integrated by quadratures, except in the last of four cases, when it is also necessary to integrate an auxiliary differential equation of the first order. But the last case is not excep-

tional, and also requires only quadratures. The oversight was due (p. 424) to not making full use of the hypothesis $(U_1U_2) \equiv U_1f$ of that case. In fact, the long Chapter 18 can be compressed into two pages by noting that under the former hypothesis or the remaining one $(U_1U_2) \equiv 0$, the infinitesimal transformation U_2f leaves invariant the partial differential equation $U_1f = 0$, whence a solution ϕ of the latter may be found by quadratures. When ϕ and y are taken as new variables, U_1f becomes $F\partial f/\partial y$, which is reduced to $\partial f/\partial y_1$ by choice of a function y_1 of ϕ and y found by an obvious quadrature. It is now a simple matter to obtain the four types of canonical forms of U_1, U_2 by quadratures only.

39. Professor L. E. Dickson: *Present status of the history of the theory of numbers.*

The purpose of this preliminary communication is to announce (1) that the third volume of the writer's *History of the Theory of Numbers*, entitled *Arithmetical theories of quadratic and higher forms*, is being put in type for the Carnegie Institution of Washington, and (2) that the publication will soon be undertaken by the National Research Council of a report prepared by Professors Dickson, H. H. Mitchell, Vandiver and Wahlin on algebraic numbers and related topics, chiefly from 1894 to date, with the primary aim to supplement the report by Hilbert in the yearbook of the German Mathematical Society for 1894.

40. Professor W. L. Hart: *The determination of a seasonal variation.*

Let $y = f(t)$ represent a statistical table, in which $t = 0, 1, \cdots, 12k$ months, where $t = 0$ is January, first year, $t = 15$ is March, second year, etc. The method of monthly means for determining the seasonal variation present in the function $f(t)$ consists of forming a January entry equal to the arithmetic mean of all the given January values, $\cdots$, a December entry equal to the mean of the December values. The author states certain useful properties of this method. He also gives an obvious justification of its use in the case of economic data, affected by the business cycle, by virtue of a certain well known property of the trigonometric functions. Attention is called to the relatively complicated character of the method for determining seasonal variation which has been used by Professor W. M. Persons, and an example is considered which illustrates the possible inaccuracy of his method.

41. Dr. V. D. Gokhale: *Concerning compact Kürschák fields.*

In the paper *Ueber Limesbildung und die allgemeine Körpertheorie* (JOURNAL FÜR MATHEMATIK, vol. 142), Kürschák sets up abstract fields with a modulus ("bewertete Körper"). This modulus plays, in the general field, essentially the same rôle as the absolute value in the fields of classical analysis, viz., real number system, complex number system, etc. Kürschák proves that every abstract field of this type has the (smallest) algebraically closed and perfect extension.

In the present paper the author sets up the notion *compactness.* This notion is analogous to M. Fréchét's compactness and to the J-compactness in E. H. Moore's General Analysis. It is a generalization of the following property in the point set theory: Every infinite set of points in a bounded domain has at least one condensation point. He then studies the properties of algebraically closed and compact fields and compactness under the adjunction of algebraic elements. Using Ostrowski's results, he proves the theorem that the smallest algebraically closed extension of a compact field is compact if and only if it can be obtained by adjoining a single algebraic element. The last part of the paper develops a complete existential theory of the four properties: (1) having a characteristic cther than zero, (2) algebraic closure, (3) perfection and (4) compactness. Out of the $2^4 = 16$ possibilities 11 are shown to be existent and the remaining 5 nonexistent.

42. Professor W. H. Roever: *A second mechanism for illustrating lines of force.*

Several years ago the author designed a mechanism for illustrating certain systems of lines of force and stream lines (see ZEITSCHRIFT FÜR MATHEMATIK UND PHYSIK, vol. 62 (1914), and also BULLETIN OF THE MOUNT WEATHER OBSERVATORY, vol. 6 (1914), Part 5). In the mechanism exhibited at this meeting, one of the two wheels of the former mechanism is replaced by a striped curtain which moves with uniform linear velocity just behind a uniformly rotating spoked wheel. For sufficiently rapid motions of the curtain and wheel the paths of the points of intersection of the spokes of the wheel with the stripes of the curtain become distinctly visible. These curves were shown to be identical with lines of force of the field resulting from the introduction of a source (or a sink) in an originally constant field of force.

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Secretary of the Chicago Section.

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Secretary of the Society.