KLEIN'S COLLECTED WORKS

Felix Klein: Gesammelte mathematische Abhandlungen. Vol. 1. *Liniengeometrie, Grundlegung der Geometrie, Zum Erlanger Programm.* Edited by R. Fricke and A. Ostrowski. Berlin, Julius Springer, 1921. xiii + 612 pages.

Felix Klein, born at Düsseldorf, April 25, 1849, was awarded the doctor's degree December 12, 1868, by the University of Bonn. The dissertation was directed and the examination conducted by Professor Lipschitz. At the golden jubilee of the doctorate, celebrated at Göttingen, December 12, 1918, plans were perfected for the publication of the numerous and varied mathematical contributions he made during the ensuing fifty years.

This was a particularly fitting thing to do. During this time no one exercised a greater influence on the development of mathematics in Germany than he. That that influence was not confined to Germany is attested by the long list of Americans who have taken the doctorate under his direction, and the very large number who have come from all parts of the world to hear his lectures or to participate in his Seminar. Notwithstanding the excellence of his personal contributions, and the inspiration of personal contact with him, probably his strongest characteristic was his skill as an organizer, and his capacity for cooperation.

Klein is essentially a geometer, and he began his work just in the zenith of the teaching of Clebsch, Kummer, Plücker and von Staudt; a worthy member of the group which included Lie, Darboux, Cremona, Clifford, Noether . . . How important are their achievements in the study of the more critical concepts of number, integral, limit, point set, etc., of to-day!

Of the three sub-titles indicated, the memoirs on line geometry occupy the first 240 pages. While the text is nowhere materially changed, a few corrections are introduced, indicated by square brackets, a considerable number of comments so marked are found in the footnotes, and each longer memoir or group of related memoirs is prefaced by biographical material written by the author, which add not only to the human interest, but aid in the understanding of the relations between various questions considered.

While at Bonn, Klein was Plücker's assistant, not only in the preparation of lecture notes on theoretical physics, but also in the writing of the *Line Geometry.** Klein intended to make physics his chief study, but acquired so much momentum in this work that he could not stop. At the time of Plücker's death the first part of the *Line Geometry* was practically all in print, but much of the second part was undeveloped. Plücker had started Klein on the problem of the quadratic complex, and the supervision of his progress was now taken over by Lipschitz. Plücker's treatment of the new idea was detailed and elementary, strikingly different from the style of Battaglini, who wrote on the quadratic complex shortly after Plücker's

* *Neue Geometrie des Raumes, gegründet auf die Betrachtung der geraden Linie als Raumelement.* Part I, pp. 1–226, edited by Clebsch, 1868; Part II, pp. 227–378, edited by Klein, 1869.

earlier papers had appeared.* Klein discovered that Battaglini's form was not the most general one, and presented the manuscript of a thesis in which the complex and the quadratic identity are both expressed as the sum of squares. But in the fall of 1868 Lipschitz received the proof-sheets of Weierstrass's forthcoming paper on elementary divisors, and he asked the candidate to include the various cases in his dissertation. The principles of the classification are found in the dissertation but the details of the transformation and its geometric meaning are reserved for one of Klein's own pupils later† and the numerous inaccuracies found in the interpretation have been corrected by various Italian mathematicians. As soon as his examination was completed, Klein went to Göttingen to be with Clebsch while preparing the second part of the *Line Geometry*, and remained until the end of the following summer semester. Both Clebsch and Noether were working on the problem of mapping an algebraic surface on the plane.

The following winter semester Klein spent in Berlin; there he met L. Kiepert, O. Stolz, and S. Lie. He heard no lectures, but took an active part in the Seminar conducted by Kummer and Weierstrass. Stolz had told Klein about non-euclidean geometry, and the latter soon felt that it would furnish an interpretation of Cayley's absolute measurement, which had already made a profound impression upon him. Curiously, Weierstrass emphatically rejected the whole idea, as distance was too invariant to be subjected to change of scale. In the spring, both Klein and Lie went to Paris, where they lived in adjoining rooms until the war broke out in July. Klein reported at once for duty, but was rejected on account of frail health; later he participated in relief work until he was taken ill with typhoid fever, which kept him bedfast for several weeks. Lie visited him while he was still an invalid, and together they wrote the paper on the asymptotic lines of the Kummer surface. Klein returned to Göttingen as a docent in January, 1871. He remained there until the call to Erlangen, in 1872. In the summer of 1871, Stolz also came to Göttingen.

With the exception of two titles (numbers XII, XIII) all of the papers appearing in this first part were written during these three years. These later papers were also concerned with line geometry, but the methods employed in them are essentially transcendental. In the preface it is explained how they naturally fit into the earlier category. They are *Ueber Konfigurationen, welche der Kummerschen Fläche zugleich eingeschrieben und umgeschrieben sind,* (MATHEMATISCHE ANNALEN, vol. 27 (1885)) and *Zur geometrischen Deutung des Abelschen Theorems der hyperelliptischen Integrale* (MATHEMATISCHE ANNALEN, vol. 28 (1886)). They were suggested by the dissertations of some of Klein's students, particularly of Rohn‡ and of Domsch.§ The most important contribution of Klein to

* *Intorno ai sistemi di rette di secondo grado*, ATTI DELLA ACCADEMIA DI NAPOLI, vol. 3 (1866).

† A. Weiler, *Ueber die verschiedenen Gattungen der Complexe zweiten Grades*, MATHEMATISCHE ANNALEN, vol. 7 (1872).

‡ *Betrachtungen über die Kummersche Fläche und ihren Zusammenhang mit den hyperelliptischen Functionen* $p = 2$, Munich, 1878.

§ *Die Darstellung der Fläche vierter Ordnung mit Doppelkegelschnitt durch hyperelliptische Functionen*, Leipzig, 1886.

line geometry is his system of coordinates, and its application to systems of quadratic complexes, to confocal systems, and to differential properties.

The second part of the present volume, pages 240–410, contains a number of early notes, and the long memoir of 1890, which contains a substantial harmonization of the earlier notes on measurement, the theory of groups, and the significance of transformations which leave a given configuration —space, variety, surface, curve—invariant. Although von Staudt's *Geometrie der Lage* was written in 1847, it was not noticed until 1871 that his treatment of harmonic elements, upon which the later *Beiträge* built the theory of coordinates, was essentially independent of the parallel axiom; it was Klein who made it sharply and distinctly so. The first paper is a brief summary which first appeared in the GÖTTINGER NACHRICHTEN, showing that Cayley's theory of measurement can be forcefully visualized by means of von Staudt's projective geometry. Besides these two sources the author calls attention to the help he got from the memoir of Beltrami* and particularly to the great inspiration which he found in the writings of Clifford. He states that he found the books of Salmon most useful and instructive.

In the fall of 1873 Klein attended the Bradford meeting of the British Association, and met Cayley, Sylvester, Clifford, and Ball (Sir Robert). The two long papers *Ueber die sogenannte Nicht-Euklidische Geometrie* had already appeared before the Bradford meeting was held, as the second was finished in 1872. The next paper was written after the Bradford meeting. It contains the axiom of projectivity in its present form, and also the statement that continuity can be postulated in the same manner as in metric geometry. Now follows a gap of six years, at the end of which appears the short note in which the author acknowledges the redundancy of his system of axioms necessary for the von Staudt development; it is a further analysis of the axiom of projectivity. The last paper appeared in 1890, at the end of the course on non-euclidean geometry given at Göttingen in 1889–1890. It contains a detailed account of Clifford's theory of parallels (right and left parallels), points out the difference between spherical and elliptic geometry, develops a purely projective foundation for analytic geometry, and discusses the meaning of axioms in general.

The third volume of Lie's *Theorie der Transformationengruppen* appeared in 1893. The preface contains (pp. x–xii) an admirable resumé of the growth of non-euclidean geometry and of the significance of the problem discussed in Part V, pages 393–543. It also contains several statements regarding the contributions of Klein to the development of non-euclidean geometry and of the foundations of geometry that are sadly out of keeping with the worth and the dignity of the other parts of that monumental work. In 1897 Klein was invited to report on the value of Part V of Lie's volume 3 to the Physico-Mathematical Society of the imperial University of Kasan in connection with the first award of the Lobachschevsky prize. The report covers 18 pages in the present volume. A more dignified or noble eulogy can hardly be found in critical literature than that sent by Klein to the Russian Society, proposing Lie's name. Revenge is sweet.

* *Saggio di interpretazione della geometria non euclidea*, GIORNALE DI BATTAGLINI, vol. 6 (1868).

The third part is concerned with the Erlangen Programm, but there is no sharp division between this and the preceding one. It begins with the joint note on W-curves, written by Klein and Lie, first the summary that appeared in the COMPTES RENDUS, then the full paper that was published a little later in the ANNALEN. In this paper one sees the different points of view of the two authors, one striving to express the phenomena as discrete operations, the other to define everything in terms of differential equations. It is a perfectly natural suggestion growing out of their joint work on non-euclidean geometry. The difference in point of view is well illustrated by a little incident, now mentioned in a footnote. The W-curves are defined as those having tangents cutting the triangle of reference in three points, which, with the point of contact, make a constant cross ratio. Klein was disturbed by the indeterminate form which arises when the curve has singular points at the vertices, and proposed that the paradox be investigated. (This has been done since in connection with the integrals of differential equations of the first order, by Poincaré, Vessiot, and others.) But Lie was so filled with the general theory that such things did not interest him. His reply was: "Die Kurve weiss selbst am besten, wie sie sich in singulären Punkten zu verhalten hat."

We now come to the Programm itself. It was prepared in the midst of the other early papers already mentioned, and while the author was still saturated with his work with Plücker, with Lie, with Stolz, and he was in daily contact with Clebsch while the actual paper was being written. To those who have worked a long time in algebraic geometry, most of the ideas there developed now seem almost self-evident. The idea of the group, as expressed in terms of birational transformations, the principal subgroup (Hauptgruppe), under which a surface remains invariant, the necessary restrictions on the families of curves on the surface, the change of space element, the geometric facts concerning invariants, are now the ordinary tools of the trade. But when we recall that this was written fifty years ago, that its author was twenty-three years old, that Jordan's treatise on substitution groups had hardly been out a year, that Cayley's papers on Cremona transformations, Noether's memoir on mapping and the Brill-Noether paper on algebraic curves were not then in existence, that Lie's line-sphere transformation appeared the same summer, then we realize that it was a veritable programme, which has been closely followed during the last half century. The contributions of the Italian geometers furnish a sufficiently eloquent tribute to the comprehensiveness of the plan there outlined. If we now re-read the preface of Volume III of Lie's *Theorie der Transformationengruppen*, especially page xvii, we are supplied with a great deal of food for serious reflection.

Klein remained at Erlangen five semesters, then went to Munich Easter, 1875, thence to Leipzig, October, 1880, and finally to Göttingen in 1886, at which time through his efforts Lie was made his successor at Leipzig. The first few years at Göttingen were devoted to systematizing and completing the earlier contributions, and occasionally to take a new glance at mechanics, which as a boy had interested him more than mathematics. Besides giving a course on mechanics he prepared the treatise on the theory of the top with Sommerfeld, the first volume of which appeared in 1897. In 1901 Sir Robert Ball's *Theory of Screws* was published. The Encyklopädie was undertaken in 1894, and Klein was appointed editor of the volume on mechanics. These facts account for the next paper, a criticism of Sir Robert's book. It emphasizes what Klein had always maintained, namely, the necessity of accounting for exceptional and singular cases, to which the general theory ceases to apply. By reading the Programm and the criticism together, one sees the close connection between them. To be consistent with his own scheme, the author should have everywhere omitted the differentials of the second order, but had this been done it would lead to the exclusion of what both the author of the book and of the criticism designate as the most important part of the theory, in those cases in which the first differentials are identically zero.

The remainder of the present volume consists of much later papers; they are concerned with the connection between the theory of relativity and the projective theory, as developed in the Programm, particularly the fundamental significance of Cayley's theory of measurement. The first paper formed the closing chapter of a course on analytic projective geometry given in 1909–1910, and makes an elementary approach to the Lorentz group, showing it to be included in the affinity group in space of four dimensions. The same also contains a clear summary of the argument for the various kinds of non-euclidean geometries of fewer dimensions, and how they are related by a continuous change of the space parameter (constant of curvature). After the elementary part, entirely algebraic, there follows a short discussion of the Maxwell electro-dynamic equations, in which it is shown that they are invariant when the space coordinates are changed linearly, or the time is translated, but not under such transformations as contain both space and time coordinates in the same equation.

The next paper is a letter to Hilbert, after the appearance of Hilbert's first note on the foundations of physics. It contains an alternate derivation of Hilbert's results by means of an appropriate transformation in the sense of the Programm. Then in reply Hilbert supplies the details of the analysis, and Klein shows, by means of a system of partial differential equations that even the general theory of relativity can be treated from this same standpoint, as well as the earlier special theory. Only those consequences of the differential equations have a physical sense that are invariant under the operations of the group—a property that is inherent in the Programm for any geometry. The last two papers of the volume apply the same transformations directly to the Einstein equations in the new theory of gravitation, leading to many material simplifications. In particular, the meaning of the symbols in the multiple integration is treated at length, bringing them into clear relation with the Grassmann symbolism. The tubular or cylinder-formed universe of Einstein is contrasted with that of constant curvature of de Sitter. The general transformation theory applies directly to the latter, whereas artificial modifications must be made before it applies to the case of Einstein.

The book is printed in large clear type on good paper, and is almost completely free from typographical errors.

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