

4. Write $u_k(m) = m^k$, and put

$$X'_k(m) = X_k(m)/m^k, \quad Z'_k(m) = Z_k(m)/m^k.$$

Then from the definitions of the functions,

$$X'_\mu = u_{-\mu}f, \quad Z'_\mu = u_{-\mu}F,$$

and from the associative and commutative laws,

$$u_{-\mu}F \cdot u_{-\nu}f = u_{-\mu}f \cdot u_{-\nu}F,$$

we find $Z'_\mu X'_\nu = X'_\mu Z'_\nu$, which may be written in full as follows:

$$\sum_m \frac{Z'_\mu(\delta)}{\delta^\mu} \frac{X'_\nu(d)}{d^\nu} = \sum_m \frac{X'_\mu(\delta)}{\delta^\mu} \frac{Z'_\nu(d)}{d^\nu}.$$

Multiplying this throughout by m^μ , we get (A).

THE UNIVERSITY OF WASHINGTON,
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A SEQUENCE OF POLYNOMIALS CONNECTED WITH THE n TH ROOTS OF UNITY.

BY DR. T. H. GRONWALL.

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In constructing examples of power series bounded in their circle of convergence and having specified convergence defects on the circle, it is frequently useful to consider polynomials of degree $n - 1$, such that at each of the n th roots of unity, the absolute value of the polynomial is less than or equal to a given constant M . Under these conditions, the maximum absolute value of the polynomial inside or on the unit circle is less than $4M \log n$.*

It is the purpose of this note to determine those polynomials where this maximum is as large as possible. The result may be stated in the following theorem.

THEOREM. *When the polynomial*

$$F(z) = a_0 + a_1z + \cdots + a_{n-1}z^{n-1}$$

* E. Landau, *Bemerkungen zu einer Arbeit des Herrn Carleman*, MATHEMATISCHE ZEITSCHRIFT, vol. 5 (1919), pp. 147-153.

has the property that

$$(1) \quad |F(\epsilon^\nu)| \leq 1 \quad (\nu = 0, 1, \dots, n-1; \epsilon = e^{2\pi i/n})$$

and $n > 1$, then

$$(2) \quad |F(z)| < \frac{1}{n} \sum_{\nu=0}^{n-1} \frac{1}{\sin \frac{2\nu+1}{2n} \pi}, \quad \text{for} \quad |z| \leq 1,$$

except when $F(z)$ has the form $e^{\alpha i f(\epsilon^{-k}z)}$, where α is a real number, k is an integer, and

$$(3) \quad f(z) = \sum_{\nu=0}^{n-1} \frac{(\epsilon^{-1/2}z)^\nu}{n \sin \frac{2\nu+1}{2n} \pi},$$

in which case the upper bound for $|F(z)|$ is reached when $z = \epsilon^{k+1/2}$. The polynomial $f(z)$ has all its zeros on the unit circle, one in each of the intervals between two consecutive n th roots of unity, except the interval between 1 and ϵ , which contains no zero.

The upper bound given on the right-hand side of (2) is asymptotically equal to

$$\frac{2}{\pi} \left(\log n + C + \log \frac{2}{\pi} \right) + o(1),$$

where C is Euler's constant, and $o(1)$ tends to zero as n increases indefinitely.

Let the absolute maximum of the absolute value of $|F(z)|$ for $|z| = 1$ occur between* ϵ^k and ϵ^{k+1} . If we write

$$F(z) = F_1(\epsilon^{-k}z),$$

the absolute maximum of $|F_1(z)|$ for $|z| = 1$ occurs between 1 and ϵ . By (1), we have for $\nu = 0, 1, \dots, n-1$,

$$F_1(\epsilon^\nu) = M_\nu e^{\alpha \nu i}, \quad 0 \leq M_\nu \leq 1;$$

and since

$$g(z) = 1 + z + z^2 + \dots + z^{n-1} = \frac{z^n - 1}{z - 1}$$

* If it occurs at an n th root of unity, then the maximum $|F(z)|$ is less than or equal to unity, which is less than the expression on the right in (2), each sine being less than unity, except the one corresponding to $\nu = (n-1)/2$ when n is odd.

is equal to n for $z = 1$, and is equal to zero for $z = \epsilon, \epsilon^2, \dots, \epsilon^{n-1}$, we have, by Lagrange's interpolation formula,

$$(4) \quad nF_1(z) = \sum_{\mu=0}^{n-1} M_{\mu} e^{\alpha_{\mu} i} g(\epsilon^{n-\mu} z).$$

Now

$$\begin{aligned} g(\epsilon^{n-\mu} e^{\theta i}) &= \frac{e^{n\theta i} - 1}{e^{(\theta + \frac{n-\mu}{n} \cdot 2\pi) i} - 1} \\ &= e^{(\frac{n-1}{2}\theta - \frac{n-\mu}{n}\pi) i} \frac{\sin \frac{n\theta}{2}}{\sin \frac{1}{2} \left(\theta + \frac{n-\mu}{n} \cdot 2\pi \right)}, \end{aligned}$$

and consequently, if we write $e^{\pi i/n} = \epsilon^{1/2}$,

$$(5) \quad nF_1(e^{\theta i}) = e^{(\frac{n-1}{2}\theta + \alpha_0) i} \times \left[M_0 \frac{\sin \frac{n\theta}{2}}{\sin \frac{\theta}{2}} - \sum_{\mu=1}^{n-1} M_{\mu} e^{(\alpha_{\mu} - \alpha_0) i} \epsilon^{\mu/2} \frac{\sin \frac{n\theta}{2}}{\sin \frac{1}{2} \left(\theta + \frac{n-\mu}{n} \cdot 2\pi \right)} \right].$$

For $0 < \theta < 2\pi/n$, all the sines in this formula are obviously positive, so that for any value of θ in this interval, we have

$$n |F_1(e^{\theta i})| \leq \frac{\sin \frac{n\theta}{2}}{\sin \frac{\theta}{2}} + \sum_{\mu=1}^{n-1} \frac{\sin \frac{n\theta}{2}}{\sin \frac{1}{2} \left(\theta + \frac{n-\mu}{n} \cdot 2\pi \right)},$$

where the equality sign holds when and only when we have $M_0 = M_1 = \dots = M_{n-1} = 1$ and $e^{\alpha_{\mu} i} = -\epsilon^{-\mu/2} e^{\alpha_0 i}$, that is when $F_1(z) = e^{\alpha_0 i} f_1(z)$, where, by (4) and (5),

$$(6) \quad n f_1(z) = g(z) - \sum_{\mu=1}^{n-1} \epsilon^{-\mu/2} g(\epsilon^{n-\mu} z),$$

$$(7) \quad n f_1(e^{\theta i}) = e^{\frac{n-1}{2}\theta i} \left[\frac{\sin \frac{n\theta}{2}}{\sin \frac{\theta}{2}} + \sum_{\mu=1}^{n-1} \frac{\sin \frac{n\theta}{2}}{\sin \frac{1}{2} \left(\theta + \frac{n-\mu}{n} \cdot 2\pi \right)} \right].$$

Since $g(z) = 1 + z + \dots + z^{n-1}$, the coefficient of z^ν in $nf_1(z)$ is, by (6),

$$\begin{aligned} 1 - \sum_{\mu=1}^{n-1} \epsilon^{-\mu/2} \epsilon^{\nu(n-\mu)} &= 1 - \frac{\epsilon^{-[(1/2)+\nu]} - \epsilon^{-[(1/2)+\nu]n}}{1 - \epsilon^{-[(1/2)+\nu]}} \\ &= 1 - \frac{\epsilon^{-[(1/2)+\nu]} + 1}{1 - \epsilon^{-[(1/2)+\nu]}} \\ &= -\frac{2}{\epsilon^{(1/2)+\nu} - 1} = \frac{i\epsilon^{-[(1/4)+(\nu/2)]}}{\sin \frac{2\nu+1}{2n}\pi}. \end{aligned}$$

Consequently, defining $f(z)$ by (3) and α by $\alpha = \alpha_0 + \pi/2 - \pi/(2n)$, we find $f_1(z) = i\epsilon^{-1/4}f(z)$, and since the absolute maximum of $|f(z)|$ for $|z| = 1$ occurs when all terms to the right in (3) are positive, that is when $z = \epsilon^{1/2}$ or $\theta = \pi/n$ (which is the midpoint of the interval from 0 to $2\pi/n$), it follows that the absolute maximum for $|z| = 1$ of $|F(z)|$ is less than the right-hand member of (2), unless $F_1(z) = e^{\alpha_0}f_1(z) = e^{\alpha}if(z)$, that is when $F(z) = e^{\alpha}if(\epsilon^{-k}z)$.

The zeros of $f(z)$ are evidently those of $f_1(z)$, and (7) shows that $\varphi(\theta) = e^{-(n-1)\theta}i/2f_1(e^{\theta}i)$ is real. Since, by (6), $f_1(1) = 1$, $f_1(\epsilon^\nu) = -\epsilon^{-\nu/2}$ for $\nu = 1, 2, \dots, n-1$, it follows that $\varphi(0) = 1$, and $\varphi(2\nu\pi/n) = (-1)^{\nu-1}$ for $\nu = 1, 2, \dots, n-1$, so that $\varphi(\theta)$ has an odd number of zeros in each of the intervals $2\nu\pi/n < \theta < (2\nu+2)\pi/n$, and consequently $f_1(z)$ has an odd number of zeros on the unit circle in each of the $n-1$ intervals from ϵ^ν to $\epsilon^{\nu+1}$ ($\nu = 1, 2, \dots, n-1$). But $f_1(z)$ being of degree $n-1$ has exactly $n-1$ zeros, all of which therefore lie on the unit circle, one in each of the intervals mentioned. To find the asymptotic value of the expression to the right in (2), we observe that the ν th and $(n-1-\nu)$ th terms in the sum are equal, and that for n odd, there is a middle term equal to unity. Hence, for n even or odd,

$$\frac{1}{n} \sum_{\nu=0}^{n-1} \frac{1}{\sin \frac{2\nu+1}{2n}\pi} = \frac{2}{n} \sum_{\nu=0}^{[n/2]-1} \frac{1}{\sin \frac{2\nu+1}{2n}\pi} + o\left(\frac{1}{n}\right),$$

and by the definition of a definite integral

$$\lim_{n \rightarrow \infty} \frac{\pi}{n} \sum_{\nu=0}^{[n/2]-1} \left(\frac{1}{\sin \frac{2\nu+1}{2n} \pi} - \frac{1}{\frac{2\nu+1}{2n} \pi} \right) \\ = \int_0^{\pi/2} \left(\frac{1}{\sin x} - \frac{1}{x} \right) dx = \log \frac{4}{\pi},$$

or

$$\frac{2}{n} \sum_{\nu=0}^{[n/2]-1} \frac{1}{\sin \frac{2\nu+1}{2n} \pi} = \frac{4}{\pi} \sum_{\nu=0}^{[n/2]-1} \frac{1}{2\nu+1} + \frac{2}{\pi} \log \frac{4}{\pi} + o(1).$$

Using the familiar asymptotic formula

$$\sum_{\nu=0}^m \frac{1}{2\nu+1} = \frac{1}{2} \log m + \frac{1}{2} C + o(1),$$

where C is Euler's constant, we find

$$\frac{1}{n} \sum_{\nu=0}^{n-1} \frac{1}{\sin \frac{2\nu+1}{2n} \pi} = \frac{2}{\pi} \left(\log n + C + \log \frac{2}{\pi} \right) + o(1).$$

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THE MINIMUM AREA BETWEEN A CURVE AND ITS CAUSTIC.

BY PROFESSOR PAUL R. RIDER.

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If rays from a given source of light are reflected by a curve, the envelope of the rays after reflection is called the caustic of the curve. It is an interesting problem to find the curve which connects two fixed points and which with its caustic and the rays reflected from the fixed points will enclose a minimum area. Euler* proposed and solved a similar problem

* Euler, *Methodus inveniendi lineas curvas maximi minime proprietate gaudentes* or German translation in Ostwald's *Klassiker der exakten Wissenschaften*, no. 46. See also Todhunter, *Researches in the calculus of variations*, chapter 13.