THE TWENTY-SEVENTH SUMMER MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

The twenty-seventh summer meeting and ninth colloquium of the Society were held at the University of Chicago on September 7–11, following the summer meeting of the Mathematical Association of America, which was held on Monday, September 6. The sessions of Tuesday and of Wednesday forenoon were devoted to the reading of papers; the colloquium opened on Wednesday afternoon.

The members of the Society and their guests found excellent arrangements for their entertainment at Hitchcock and Beecher Halls, and at the Quadrangle Club. The Club was very graciously put at the disposal of the members for meals and for use as a social center. On Friday afternoon visits were made under the guidance of Professor Slaught to Ida Noyes Hall and to Harper Memorial Library.

At the joint dinner of the two organizations, one hundred and sixteen persons were present. Professor Slaught, retiring president of the Mathematical Association, presided at the after-dinner speaking, which was participated in by several of those present, to the edification of all the listeners.

The meetings of the Society were attended by more than one hundred and twenty persons, among whom were the following ninety-nine members of the Society:

Professor R. C. Archibald, Professor G. N. Armstrong, Professor I. A. Barnett, Professor Suzan R. Benedict, Professor A. A. Bennett, Professor G. D. Birkhoff, Professor G. A. Bliss, Professor R. L. Borger, Professor J. W. Bradshaw, Professor W. C. Brenke, Professor W. H. Bussey, Professor W. D. Cairns, Professor J. W. Campbell, Professor A. L. Candy, Professor E. W. Chittenden, Professor C. E. Comstock, Professor A. R. Crathorne, Dr. G. H. Cresse, Professor D. R. Curtiss, Professor H. H. Dalaker, Professor S. C. Davisson, Professor E. L. Dodd, Professor L. W. Dowling, Professor Arnold Dresden, Professor Otto Dunkel, Professor M. D. Earle, Professor Arnold Emch, Professor G. C. Evans, Professor H. S. Everett, Professor Peter Field, Professor B. F. Finkel, Professor L. R. Ford, Professor W. B. Ford, Professor M. G. Gaba, Professor W. H. Garrett, Professor

D. C. Gillespie, Professor R. E. Gilman, Dr. T. H. Gronwall, Professor C. F. Gummer, Professor W. L. Hart, Professor M. W. Haskell, Professor Olive C. Hazlett, Professor E. R. Hedrick, Professor T. H. Hildebrandt, Professor L. A. Hopkins, Professor W. A. Hurwitz, Professor Dunham Jackson, Mr. D. C. Kazarinoff, Professor S. D. Killam, Professor H. W. Kuhn, Professor Gillie A. Larew, Professor Flora E. LeStourgeon, Professor A. C. Lunn, Professor E. B. Lytle, Professor R. B. McClenon, Professor J. V. McKelvey, Professor T. E. Mason, Professor Helen A. Merrill, Professor Bessie I. Miller, Professor W. L. Miser, Professor C. N. Moore, Professor E. H. Moore, Professor E. J. Moulton, Professor F. R. Moulton, Professor A. L. Nelson, Professor Anna H. Palmié, Professor Anna J. Pell, Dr. T. A. Pierce, Professor A. D. Pitcher, Professor S. E. Rasor, Professor R. G. D. Richardson, Professor H. L. Rietz, Professor Maria M. Roberts, Professor W. H. Roever, Professor Oscar Schmiedel, Dr. Caroline E. Seely, Professor E. W. Sheldon, Dr. W. G. Simon, Professor E. B. Skinner, Professor H. E. Slaught, Mr. H. L. Smith, Professor E. B. Stouffer, Professor C. E. Stromquist, Professor K. D. Swartzel, Professor E. J. Townsend, Professor A. L. Underhill, Professor E. B. Van Vleck, Professor Oswald Veblen, Dr. J. H. Weaver, Professor W. P. Webber, Mr. F. M. Weida, Professor Mary E. Wells, Professor W. D. A. Westfall, Professor Marion B. White, Professor C. E. Wilder, Professor F. B. Wiley, Professor C. H. Yeaton, Professor J. W. Young, Professor J. W. A. Young.

Vice-President R. G. D. Richardson presided at the meetings of Tuesday and Wednesday forenoons; Professor Haskell presided on Tuesday afternoon.

Upon the recommendation of the Council the Society adopted the following amendments to the By-Laws:

1. By-Law II, section 2, is amended by changing the words "five dollars" to "six dollars," so that the amended section reads "The annual dues shall be six dollars payable on the first of January, etc." Adopted unanimously.

2. By-Law II, section 3, is amended by changing the words "fifty dollars" to "seventy-five dollars," so that the amended section reads "On the payment of seventy-five dollars in one sum, any member of at least four years standing and not in arrears of dues may become a life member and shall thereafter be exempt from all annual dues." Adopted by a vote of 47 to 2.

3. By-Law IX, section 2 is amended by the addition of the sentence: "One member of the Committee of Publication shall be designated by the Council as Editor of the BULLETIN." Adopted unanimously.

The Council announced the election of the following persons to membership in the Society: Dr. R. F. Borden, Brown University; Dr. Tso Chiang, Nan Kai College, Tientsin, China; Professor H. M. Dadourian, Trinity College, Hartford, Conn.; Dr. Jesse Douglas, Columbia University; Mr. Philip Franklin, Princeton University; Dr. C. F. Green, University of Illinois; Captain R. S. Hoar, Ordnance School, Aberdeen, Md.; Dr. Jessie M. Jacobs, University of Texas; Dr. E. L. Post, Princeton University; Professor C. D. Rice, University of Texas; Mr. L. G. Simon, New York City; Professor J. E. Stocker, Lehigh University; Dr. Tsao-Shing Yang, Syracuse University. Twenty-two applications for membership in the Society were received.

The following resolution was adopted unanimously: "The American Mathematical Society expresses to the department of mathematics of the University of Chicago its heartiest appreciation of the arrangements they have made for the twenty-seventh summer meeting of the Society and its gratitude for their hospitality."

Professor Hedrick spoke on behalf of the Committee on increase of membership and of sales of the Society's publications. A considerable number of new subscriptions to the *Transactions* were secured subsequently.

Professor Bliss then offered the following resolution, which was adopted unanimously: "The Society recommends for favorable consideration by the Council applications for membership from advanced students and others interested in mathematics, whether engaged in teaching or not, when properly proposed by members of the Society."

The following papers were read at this meeting:

(1) Professor ARNOLD EMCH: "On the projective generation of cyclides."

(2) Dr. J. H. WEAVER: "A generalization of the strophoid."

(3) Professor C. F. GUMMER: "On the relative distribution of the real roots of two real polynomials."

(4) Professor A. A. BENNETT: "The polyadic expansion of a number."

(5) Dr. J. L. WALSH: "On the location of the roots of the jacobian of two binary forms."

(6) Dr. J. L. Walsh: "On the transformation of convex point sets."

(7) Professor W. B. Ford: "On Kakeya's minimum area problem."

(8) Professor T. H. Hildebrandt: "On completely continuous linear transformations."

(9) Professor Anna J. Pell: "Integral equations in which the kernel is quadratic in the parameter."

(10) Professor Olive C. Hazlett: "Annihilators of modular invariants."

(11) Professors Virgil Snyder and F. R. Sharpe: "Construction of multiple correspondences between two algebraic curves."

(12) Professor Dunham Jackson: "Note on a method of proof in the theory of Fourier's series."

(13) Professor J. W. Campbell: "On the drift of spinning projectiles."

(14) Professor W. L. Hart: "Functions of infinitely many variables in Hilbert space."

(15) Professor D. C. Gillespie: "A property of continuity."

(16) Professor L. A. Hopkins: "Periodic orbits of type 2/1."

(17) Professor Dunham Jackson: "Note on the median of a set of numbers."

(18) Professor C. N. Moore: "An application to Weierstrass's function of the generalized derivative of type $(C\,1)$."

(19) Dr. L. R. Ford: "A method of graduating curves."

(20) Professor E. W. Chittenden: "Note on a generalization of a theorem of Baire."

(21) Professor E. W. Chittenden: "On classes of functions defined in terms of relatively uniform convergence."

(22) Professor E. W. Chittenden: "On the relation between the Hilbert space and the calcul fonctionnel of Fréchet."

(23) Dr. J. L. Walsh: "A generalization of the Fourier cosine series."

(24) Professor Dunham Jackson: "Note on a class of polynomials of approximation."

(25) Professor G. A. Miller: "Reciprocal subgroups of an abelian group."

(26) Professors E. R. Hedrick, W. D. A. Westfall and Louis Ingold: "Characteristic lines of transformations."

(27) Professor W. L. HART: "Pseudo-differentiation of a summable function with respect to a parameter."

(28) Professor EDWARD KASNER: "Five notes on Einstein's theory of gravitation."

(29) Professor DUNHAM JACKSON: "On the convergence of certain trigonometric approximations."

(30) Professor DUNHAM JACKSON: "Note on the Picard method of successive approximations."

(31) Professor OLIVE C. HAZLETT: "A symbolic notation in the theory of formal modular invariants."

(32) Dr. T. H. GRONWALL: "On the Fourier coefficients of a continuous function."

(33) Dr. T. H. GRONWALL: "A sequence of polynomials connected with the nth roots of unity."

(34) Dr. T. H. GRONWALL: "Upper bounds for the coefficients in conformal mapping."

The papers of Professors Snyder and Sharpe, Dr. Walsh, Professor Miller, Professor Kasner, Professor Chittenden's first paper, Professor Jackson's fifth paper and Dr. Gronwall's first and second papers were read by title. Abstracts of the papers numbered in accordance with the above list of titles are given below.

1. Professor Emch shows how cyclides may be treated from the projective point of view. In the first place it is proved that every cyclide may be generated in an infinite number of ways by two projective pencils of spheres (of which one may degenerate into a pencil of planes). By this method of generation many of the well-known properties of cyclides may easily be derived. It also leads to a generalized cyclide which may be called a quintic cyclide, whose definition is contained in the following theorem:

The tangent planes of a quadric cone and an independent projective pencil of spheres generate a quintic cyclide with a finite circle as a double curve. This quintic degenerates into an ordinary general cyclide when the radical plane of the pencil of spheres coincides with the projectively corresponding tangent plane of the cone.

When α, β, γ are linear quantics representing planes, P and Q quadratic quantics representing spheres, and λ denotes a parameter, the cone of class 2 and the projective pencil of spheres have the form

$$\alpha\lambda^2 + \beta\lambda + \gamma = 0, \quad P - \lambda Q = 0,$$

and the quintic cyclide is

$$\alpha P^2 + \beta PQ + \gamma Q^2 = 0.$$

2. Among the generalizations of the strophoid given by Loria in his treatise Algebraische und transcendente ebene Kurven, the one by W. W. Johnson is as follows: Let there be two fixed points A and B and let two lines l_1 and l_2 make angles φ_1 and φ_2 with the line AB such that

$$m\varphi_1 \pm n\varphi_2 = \alpha \quad (\alpha = \text{constant}).$$

Then if l_1 and l_2 intersect in P, the locus of P is a strophoid.

Since α is a constant there is associated with the strophoid a a circle passing through A and B. Dr. Weaver has developed strophoids having associated with them the three conic sections in the same relation that the circle is associated with the strophoid of Johnson, and has proved a number of theorems on concurrence of lines and collinearity of points connected with the curves.

3. The roots of a real polynomial which lie within a given interval on the real axis are divided into groups by the roots of a second real polynomial. Professor Gummer shows how this distribution may be determined by a rational process. A generating function $\Sigma_r P_r(1 + t)^{k-r}(1 - t)^r$ is formed such that the coefficients of descending powers of t are the numbers in the successive groups from left to right. Two methods are given for the determination of the P's, corresponding to the theorems of Sturm and Hermite for a single polynomial. Sylvester had given a method for the reduced arrangement in which pairs of roots of either polynomial occurring consecutively were omitted.

4. In this paper Professor Bennett discusses the array consisting of all the p-adic expansions of a given integer, as p ranges through the sequence of primes. For a given p, the p-adic coefficients constitute a row of the array and for a given n, as p varies, the coefficients of p^n constitute a column. For every integer, positive or negative, there is an array, or polyadic expansion, while a polyadic expansion does not in general represent an actual integer. Sections of this array form the natural instruments in the study of numerical modular domains with a composite modulus. Subtraction and division are considered.

5. An annular region is defined as a closed region of the plane bounded by two non-intersecting circles. Dr. Walsh proves that if three annular regions are respectively the envelopes of three points z_1, z_2, z_3, then the envelope of the point z_4 defined by the real constant cross ratio

$$\lambda = (z_1, z_2, z_3, z_4)$$

is also an annular region. This result is applied in giving some geometric results concerning the location of the roots of the jacobian of two binary forms.

If f_1 and f_2 are binary forms of respective degrees p_1 and p_2, then the roots of the jacobian of f_1 and f_2 are the real foci of a certain curve of class $p_1 + p_2 - 1$ which touches all the lines formed by joining the pairs of roots of f_1 and f_2 in all possible ways. The curve has various other interesting properties.

6. The chief purpose of Dr. Walsh's note is to prove that a necessary and sufficient condition that a one-to-one point transformation of the plane or of space transform every convex point set into a convex point set is that it be a collineation.

7. Kakeya's problem in its simplest form is as follows: How should a line segment AB be turned end for end in a plane so as to sweep out a minimum area? Professor Ford shows that there are an infinite number of methods of attaining such a minimum area if it be assumed that, as the segment moves, the area generated nowhere returns into itself; but if it be assumed that it returns into itself exactly once in one or more regions, that is, if duplication be allowed, then there is no minimum area attainable by motions confined to the finite plane.

8. In a paper entitled "Lineare Funktionalgleichungen"* F. Riesz has discussed the linear integral equation in an elegant manner, which on account of its fundamental character seems destined to point the way to new and more general results in allied fields. In the first part of Professor Hildebrandt's paper a simple general basis is provided in which the methods and results of Riesz are valid. The second part of the paper takes up a more detailed study of the inversion of the com-

* *Acta Mathematica*, vol. 41 (1916), pp. 71–96.

pletely continuous transformation which leads to a generalization of the notion of the solution of adjoint homogenous integral equations, and of pseudo-resolvents.

9. Mrs. Pell considers the questions of the existence of solutions of linear integral equations in which the kernel has the form $\lambda K(x, s) + \lambda^2 L(x, s)$ and K and L satisfy certain conditions, and the expansion of arbitrary functions in terms of the solutions.

10. In any theory of invariants, differential operators which annihilate invariants are useful in the computation of invariants and covariants, and are sometimes important in the development of the general theory. Professor Hazlett's paper determines such annihilators for modular invariants. These operators are in some ways analogous to the well known Aronhold operators in the theory of classical invariants, but are very much more complicated, as might be expected. They are, in fact, of the general type anticipated by Professor Dickson in a paper published in 1907.

11. It is well known (Hurwitz, *Mathematische Annalen*, volume 28) that algebraic correspondences between two algebraic curves exist which require two auxiliary equations for their definition. The condition that two equations shall be required was found by Castelnuovo (*Rendiconti dei Lincei*, 1906), but no illustrations have been given. The example given by Amodeo (*Annali*, series 2, volume 20), cited by Castelnuovo, of the intersection of two ruled surfaces in general position can be defined by one equation. The paper by Professors Snyder and Sharpe discusses this example, and gives various methods of constructing correspondences that require two equations for their definition. The paper will appear in the *Transactions*.

12. Professor Jackson's first paper will appear in a later number of the BULLETIN.

13. Professor Campbell treats the problem of drift for the case in which initially the rotation of the projectile is entirely about the axis. He obtains a formula which exhibits qualitatively the well-known characteristics of the phenomenon. An application to the British Mark VI gives results which are quantitatively consistent with observed values.

14. In considering functions of infinitely many real variables $(x_1, x_2, x_3, \ldots)$, the case which is of most interest as viewed from the standpoint of the theory of integral equations and of more general functional equations is that in which it is supposed that $\Sigma_{i=1}^{\infty} x_i^2$ converges. In the present paper, Professor Hart assumes that the points $\xi = (x_1, x_2, \cdots)$ considered satisfy this condition.

A function $f(\xi)$ is said to be completely continuous at a point $\xi = (a_1, a_2, \cdots)$ if the second of the following equations holds whenever the first does:

$$\lim_{n=\infty} \sum_{j=1}^{\infty} (x_{jn} - a_j)^2 = 0; \tag{1}$$

$$\lim_{n=\infty} f(x_{1n}, x_{2n}, \cdots) = f(a_1, a_2, \cdots). \tag{2}$$

Professor Hart first proves certain general theorems, regarding completely continuous functions $f(\xi)$, including a mean value theorem which gives, as a corollary, an expression for the differential of $f(\xi)$. There is then considered the proof of the existence of a continuous solution of the infinite system of equations

$$f_i(t, x_1, x_2, \cdots) = 0 \qquad (i = 1, 2, \cdots), \tag{3}$$

and the existence of a solution of the system of differential equations

$$\frac{dx_i}{dt} = g_i(t, x_1, x_2, \cdots) \qquad (i = 1, 2, \cdots). \tag{4}$$

In (3) and (4) the f_i and g_i are supposed to be completely continuous in their arguments.

15. A continuous function $f(x)$ has the property that between x_1 and x_2 there is at least one value of x for which $f(x)$ has any prescribed value between $f(x_1)$ and $f(x_2)$. This property, common to continuous functions, is possessed by some discontinuous functions. Professor Gillespie's note is concerned with functions which have this property. Conditions which are sufficient to insure their continuity are developed; the character of the discontinuities that such a function may have is shown; and a function, having this property and having its set of points of continuity and its set of points of discontinuity each everywhere dense, is constructed.

16. By combining the analytic processes of Poincaré with mechanical quadrature, Professor Hopkins has obtained a family of periodic orbits in the restricted problem of three bodies, which has significance in the study of the gap in the asteroids where the period would be one half the period of Jupiter, and in the consideration of Cassini's gap in the rings of Saturn. These orbits constitute an exceptional case in certain work of F. R. Moulton and are related to results of Poincaré, E. W. Brown and G. W. Hill.

17. Professor Jackson's second paper will appear in a later number of the BULLETIN.

18. In this paper Professor Moore shows that under certain conditions Weierstrass's function, $f(x) = \Sigma a^n \cos b^n \pi x$, whose derivative in the ordinary sense fails to exist, has a generalized derivative of type $(C1)$. The definition of such generalized derivatives has been given in two papers previously presented to the Society. (Cf. this BULLETIN, volume 25 (1919), pages 249 and 257.)

19. In the smoothing of irregular curves based on original data a satisfactory combination of smoothness and faithfulness to the data is desired. Professor Ford considers the case of a function $f(x)$, given for equidistant values of the argument $x = 0, 1, \cdots, m$, and determines the graduated function $y(x)$ by minimizing the sum

$$\sum_{x=0}^{m} \{M(x)[y(x) - f(x)]^2 + [\Delta^n y(x)]^2\},$$

where the weight $M(x)$ is positive, and where n is the order of differences it is desired to make as small as possible. This leads to reasonably simple numerical methods, and, if $M(x)$ is constant, the solution has the property, desirable in certain cases of graduation, that its moments, up to the $(n - 1)$th, about any point are the same as the corresponding moments of the ungraduated function.

20. Professor Chittenden's first paper appeared in full in the October number of the BULLETIN.

21. Denote by B_0 the class of all continuous functions defined on a limited perfect subset P of n-space; by B_α the class

of all functions which do not belong to any class of order less than α and are limits of sequences of functions of class less than α. The theory of the ordinal series of classes B_α has been developed by Baire, Lebesgue and Vallée-Poussin. In terms of the relatively uniform convergence of E. H. Moore, Professor Chittenden has previously defined the corresponding series of classes R_α and now reports on the results of further investigations.

A set E is of type (α, α) at most if its characteristic function is of class B_α at most. Denote by A_α the class of all functions of class B_α such that for every pair of numbers a, b the set $E = [a < f < b]$ is of type (α, α) at most; and by F_α the set of all functions of class B_α which assume a limited number of distinct values. Then $F_\alpha < A_\alpha$ and we have $F_{\alpha+1} < R_{\alpha+1} \leqq A_{\alpha+1} + B_\alpha$, while if α is of the second kind, then $F_\alpha \leqq R_\alpha \leqq A_\alpha < B_\alpha$. Necessary and sufficient conditions that a function be of class R_α are found; but, except in the case $\alpha \leqq 1$, it has not been determined whether or not A_α contains functions not in R_α. The classes F_α and A_α are investigated, with results of interest in the general theory of the series of classes B_α.

22. It is well known that the points of the Hilbert space Ω of infinitely many dimensions can be placed in one-to-one correspondence with the set Ω_1 of all summable functions on an interval I, if we agree to identify functions of Ω_1 which differ only on a set of points of measure zero. Fréchet has called attention to the fact that the subset φ of Ω which corresponds to the set of all continuous functions on I is not closed under the usual definition of distance for the Hilbert space and suggests the desirability of a definition of distance for that space relative to which the set φ will be closed.

Employing results of Féjer and the theory of convergence in mean, Professor Chittenden defines distance for the Hilbert space in terms of the coordinates of a point and trigonometric functions so that limit in the Hilbert space becomes equivalent to uniform convergence on I, excepting a set of points of measure zero. When the functions considered are continuous this implies the continuity of the limit function and leads to the desired closure of the set φ.

The paper will be published in the *Palermo Rendiconti.*

23. Dr. Walsh proves by use of the theory of infinitely many variables that if we consider the set of functions

$$\frac{1}{\pi}\cos\lambda_0 x,\quad \frac{2}{\pi}\cos\lambda_1 x,\quad \frac{2}{\pi}\cos\lambda_2 x,\quad \frac{2}{\pi}\cos\lambda_3 x,\ \cdots \quad (0 \leqq x \leqq \pi),$$

where

$$\lambda_0^2 + 4(\lambda_1 - 1)^2 + 4(\lambda_2 - 2)^2 + 4(\lambda_3 - 3)^2 + \cdots < \frac{1}{\pi}$$

and where

$$\sum_{n=1}^{\infty} n^2 |\lambda_n - n|$$

converges, then there exists another set of functions $\{v_n(x)\}$ biorthogonal to that set. Moreover, for any function $f(x)$ integrable in the sense of Lebesgue and with an integrable square, the two series

$$\tfrac{1}{2}a_0 + a_1\cos x + a_2\cos 2x + \cdots,\quad a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx dx,$$

$$\frac{b_0}{\pi}\cos\lambda_0 x + \frac{2b_1}{\pi}\cos\lambda_1 x + \frac{2b_2}{\pi}\cos\lambda_2 x + \cdots,$$

$$b_n = \int_0^{\pi} f(x) v_n(x) dx,$$

have essentially the same convergence properties throughout the interval $0 \leqq x \leqq \pi$.

24. In a paper recently presented to the Society (see this BULLETIN, June, 1920, page 391) it was shown that if $f(x)$ is a given continuous function in the interval $a \leqq x \leqq b$, n a given positive integer, and m a given real number greater than 1, there exists one and just one polynomial $\varphi(x)$, of degree n or lower, which reduces to a minimum the value of the integral

$$\int_a^b |f(x) - \varphi(x)|^m dx.$$

The purpose of Professor Jackson's third paper is to establish the truth of the same proposition in the case that $m = 1$. The existence of at least one minimizing polynomial is proved in essentially the same way as before, the details of the argument being somewhat simpler in the present case. The proof of uniqueness is considerably less direct, and appears to involve elementary considerations of the measure of point sets. A notable difference between the two cases is that for $m > 1$ the remainder corresponding to the polynomial of best approximation must change sign at least $n + 1$ times in the interval, unless it vanishes identically, while for $m = 1$ the assertion is merely that the remainder must change sign at least $n + 1$ times, or else vanish throughout a finite number or an enumerable infinity of intervals contained in (a, b), and having at least a certain specified aggregate length.

25. Any two subgroups of the group G which have the property that the product of their orders is equal to the order of G have been called reciprocal subgroups of G. In the present paper Professor Miller confines his attention to reciprocal subgroups of abelian groups. Among the theorems proved are the following: If two reciprocal subgroups of any abelian group have only identity in common, the number of the conjugates of the one under the group of isomorphisms is equal to the number of the conjugates of the other under this group. Whenever an abelian group of order p^m, p being a prime number, contains subgroups of the same order but of different types, then the number of the subgroups of one and of only one of these types is of the form $1 + kp$. The number of the subgroups of each of the other types is divisible by p. A necessary and sufficient condition that the number of the subgroups of a given type contained in such a group is a power of p is that the number of its independent generators of each order increased by the number of the larger independent generators in the set is equal to the number of the independent generators of G whose orders are not less than this order.

26. In this paper Professors Hedrick, Westfall, and Ingold discuss the properties of a double orthogonal set of lines defined for any transformation that is not conformal by the property that their directions at each point are the directions of maximum and minimum stretching due to the transformation. These lines were originally defined by Tissot by a different property. It is shown that there always exists a transformation which corresponds to any preassigned set of such characteristic lines, and that the rate of stretching along one family may also be preassigned. Other properties are found and special cases are discussed.

27. Consider a function $u(s, t)$, where $a \leqq s \leqq b$ and where t assumes all values in a measurable set E. Let us suppose that, for every value of s, $u(s, t)$ becomes a function of t which, together with its square $u^2(s, t)$, is summable in the Lebesgue sense in E. Professor Hart defines $u_s(s_0, t)$ as the pseudo-derivative function of $u(s, t)$ with respect to s at the point s_0 if it satisfies the condition

$$\lim_{\Delta s=0}\int_E \left[\frac{u(s_0 + \Delta s, t) - u(s_0, t)}{\Delta s} - u_s(s_0, t)\right]^2 dt = 0.$$

If $u_s(s_0, t)$ exists, it is unique except for its values at a set of points t of measure zero. If $u_s(s, t)$ exists and if $g(t)$ and $(g(t))^2$ are summable, the function

$$h(s) = \int_E u(s, t)g(t)dt$$

has a derivative dh/ds. It is found that the Fourier constants $[x_1(s), x_2(s), \cdots]$ and $[y_1(s), y_2(s), \cdots]$ of $u(s, t)$ and $u_s(s, t)$, respectively, relative to a complete, unitary orthogonal system of functions $[g_1(t), g_2(t), \cdots]$, satisfy the relations

$$dx_i(s)/ds = y_i(s) \qquad (i = 1, 2, \cdots).$$

Under suitable conditions regarding the summability of the function $u_s(s, t)$ in the rectangle $a \leqq s \leqq b$, $c \leqq t \leqq d$, there is given a generalization of the mean value theorem for the function $u(s, t)$. After a few auxiliary results are established, the solution of a certain type of pseudo-differential functional equations is obtained by means of certain theorems regarding differential equations in infinitely many variables.

28. Professor Kasner's notes all relate to four-dimensional riemannian manifolds $ds^2 = \Sigma g_{ij}dx_idx_j$ obeying Einstein's ten gravitational equations $R_{ij} = 0$, where the g's are the ten potentials and the equations denote the vanishing of the so-called contracted Riemann-Christoffel tensor (which might appropriately be called the Einstein tensor). The main results follow:

(1) It is known that if the paths of particles *and* light pulses can be regarded (in some coordinate system) as straight lines, then the ds^2 is necessarily equivalent to the euclidean form. The same result is here shown to follow if the hypothesis is limited to particles alone, or to light pulses alone.

(2) Can two Einstein manifolds ever have the same light paths? This is shown to be impossible, at least in the case of approximately euclidean manifolds. Application to the one-body problem (solar gravitation): the Mercury effect could be predicted from the observed light deflection.

(3) Approximately euclidean solutions which can be expressed as the sum of four squares; in particular the quasi-conformal type $A(dx_1^2 + dx_2^2 + dx_3^2) + Bdx_4^2$ (which includes the solar field) in explicit form.

(4) If the coefficients of the four squares are functions of a single variable, the only exact (finite) solutions are certain exponentials or powers.

(5) Discussion of the possible Einstein manifolds which can be regarded as immersed in a flat space of 5 or 6 dimensions. The solar field appears in the latter case.

29. This paper is concerned with the trigonometric sums $T_{mn}(x)$ which furnish the closest approximation to a given continuous periodic function $f(x)$, in the sense of the integral of the mth power of the absolute value of the error, for prescribed values of the exponent m and the order n of the sum. The question at issue is the convergence of $T_{mn}(x)$ to the value $f(x)$, when m is held fast ($m \geqq 1$) and n is allowed to become infinite. Professor Jackson shows that a sufficient condition for uniform convergence is that $\lim_{\delta=0} \omega(\delta)/\sqrt[m]{\delta} = 0$, where $\omega(\delta)$ is the maximum of $|f(x') - f(x'')|$ for $|x' - x''| \leqq \delta$. For $m = 1$, it is sufficient that $f(x)$ have a continuous derivative. The proof makes use of Bernstein's theorem on the derivative of a trigonometric sum. As the condition obtained is less general, even for large values of m, than the well-known Lipschitz-Dini condition in the case of Fourier's series, $m = 2$, the present results may be regarded as preliminary.

30. In presenting the Picard existence proof for a differential equation $dy/dx = f(x, y)$, it is customary to assume that f is continuous, and satisfies a Lipschitz condition in y, throughout a certain rectangle in the xy-plane, and then to show that the successive approximations converge to a solution of the differential equation throughout a sufficiently restricted interval of values of x. Professor Jackson points out that by extending the definition of f outside the rectangle, so that the continuity conditions are maintained, but otherwise

arbitrarily, the process can be made to converge throughout the entire range of values of x originally considered. A function is obtained which satisfies the *original* differential equation as long as the x and y *of the solution* remain in the rectangle, whatever the behavior of the approximating functions may be. This remark is rather obvious, but it is conspicuously omitted from standard treatments of the subject. It applies equally well to systems of n differential equations in n unknown functions, and to regions that are not rectangular.

31. Professor Hazlett's second paper considers a symbolic notation for the modular invariants of a binary form with respect to the general Galois field $GF[p^n]$, of order p^n. In the classic theory of algebraic invariants, a convenient symbolic notation is obtained by expressing the binary form f of order n symbolically as a perfect nth power, $f = \alpha_x^n = \beta_x^n = \gamma_x^n = \cdots$, where $\alpha_x = \alpha_1 x_1 + \alpha_2 x_2$, $\beta_x = \beta_1 x_1 + \beta_2 x_2$, etc. Such a notation is not, however, practicable in the theory of modular invariants. If we express f as the product of n linear factors which are symbolically distinct, $f = \alpha_x \beta_x \cdots$, we have a symbolic notation by the aid of which we can write down all modular invariants of f, both formal and otherwise. In fact every formal modular invariant can be expressed as a polynomial in a finite number of symbolic expressions which behave like invariants. This theorem is illustrated for the binary cubic modulo 2 and the binary quadratic modulo 3. There is a systematic method by which any formal modular invariant for these two special cases can be expressed in symbolic form, and the writer suspects that such is true in general.

32. Given any real and single valued function $\varphi(x)$ which tends to infinity with x, Dr. Gronwall shows how to construct a function $f(\theta)$ continuous for $0 \leqq \theta \leqq 2\pi$ and such that its Fourier coefficients a_n, b_n (which have the well-known property that $\Sigma(a_n^2 + b_n^2)$ converges) make the series

$$\Sigma(a_n^2 + b_n^2)\varphi\left(\frac{1}{a_n^2 + b_n^2}\right)$$

divergent.

33. Dr. Gronwall considers a polynomial $F(z)$ of degree $n - 1$ which does not exceed unity in absolute value at the nth roots of unity, so that $|F(z)| \leqq 1$ for $z = 1, \epsilon, \epsilon^2, \cdots, \epsilon^{n-1}$

where $\epsilon = e^{2\pi i/n}$. It is shown that on the unit circle $|z| = 1$ we have

$$(1) \qquad |F(z)| < \frac{1}{n}\sum_{\nu=0}^{n-1} \frac{1}{\sin \frac{2\nu + 1}{2n}\pi}$$

except when $F(z)$ has the form $e^{\alpha i}f(\epsilon^{-k}z)$, where α is real, k an integer and

$$(2) \qquad f(z) = \sum_{\nu=0}^{n-1} \frac{(\epsilon^{-\frac{1}{2}}z)^{\nu}}{n \sin \frac{2\nu + 1}{2n}\pi},$$

in which case the upper bound of $|F(z)|$ is reached at $z = \epsilon^{\frac{1}{2}+k}$. The polynomial $f(z)$ has all its zeros on the unit circle, one in each of the intervals from ϵ to ϵ^2, ϵ^2 to ϵ^3, $\cdots$, ϵ^{n-1} to 1. The asymptotic value of the upper bound in (1) is

$$\frac{2}{\pi}\left(\log n + C + \log \frac{2}{\pi}\right) + \sigma(1)$$

where C is Euler's constant, and $\sigma(1)$ tends to zero as n increases indefinitely.

34. Dr. Gronwall shows that when $w = z + a_2z^2 + \cdots a_nz^n + \cdots$ maps the circle $|z| < 1$ conformally on a simply connected and nowhere overlapping region in the w-plane, then $|a_n| \leqq n$ for $n = 2, 3, \cdots$. When $|a_n| = n$ for any particular n, then also $|a_n| = n$ for every n, and the function w reduces to the one which gives extreme values to the distortion.

ARNOLD DRESDEN,
Acting Secretary.

THE CHICAGO COLLOQUIUM.

THE ninth colloquium of the American Mathematical Society was held in connection with its twenty-seventh summer meeting at the University of Chicago, September 8–11, 1920. At the annual meeting of 1917, the Council, on the invitation of the Department of Mathematics of the University of Chicago, appointed a committee, consisting of Professors E. H. Moore, E. W. Brown, Max Mason, H. S. White, and the Secretary, to arrange for a summer meeting and