

THE FOURTEENTH WESTERN MEETING OF THE
AMERICAN MATHEMATICAL SOCIETY.

THE fourteenth regular western meeting of the American Mathematical Society, being the forty-fifth regular meeting of the Chicago Section, was held at the University of Chicago on Friday and Saturday, April 9 and 10, 1920. The total attendance at this meeting was one of the largest in the history of the Chicago Section, there being about one hundred and fifty persons present at the meeting of Friday afternoon. Among these were the following sixty-nine members of the Society:

Professor G. A. Bliss, Professor H. Blumberg, Professor R. L. Borger, Professor W. C. Brenke, Professor E. W. Brown, Professor W. D. Cairns, Dr. C. C. Camp, Dr. J. W. Campbell, Professor J. A. Caparo, Professor R. D. Carmichael, Professor A. B. Coble, Professor A. R. Crathorne, Mr. H. W. Curjel, Dr. H. B. Curtis, Professor D. R. Curtiss, Professor S. C. Davisson, Dr. W. W. Denton, Professor L. E. Dickson, Professor Arnold Dresden, Professor Arnold Emch, Mr. E. B. Escott, Mr. T. C. Fry, Professor M. E. Graber, Professor Harris Hancock, Professor W. L. Hart, Professor E. R. Hedrick, Professor T. H. Hildebrandt, Professor G. O. James, Professor A. M. Kenyon, Dr. J. R. Kline, Professor W. C. Krathwohl, Professor J. W. Lasley, Jr., Professor Kurt Laves, Professor A. C. Lunn, Professor J. V. McKelvey, Professor Malcolm McNeill, Professor W. D. MacMillan, Professor Max Mason, Professor T. E. Mason, Professor Bessie I. Miller, Professor W. L. Miser, Professor E. H. Moore, Professor E. E. Moots, Professor E. J. Moulton, Professor F. R. Moulton, Professor G. W. Myers, Professor C. I. Palmer, Professor A. D. Pitcher, Professor S. E. Razor, Professor P. R. Rider, Professor H. L. Rietz, Professor W. J. Risley, Professor W. H. Roever, Professor D. A. Rothrock, Professor Oscar Schmiedel, Professor G. T. Sellew, Professor E. B. Skinner, Professor H. E. Slaughter, Professor R. B. Stone, Professor E. J. Townsend, Professor A. L. Underhill, Professor E. J. Wilczynski, Professor C. E. Wilder, Professor K. P. Williams, Professor R. E. Wilson, Professor C. H. Yeaton, Professor A. E. Young, Professor J. W. A. Young, Professor Alexander Ziwet.

On Friday evening seventy-four members and guests gathered at a dinner at the Quadrangle Club.

The session of Friday afternoon, presided over by Professors E. W. Brown and D. R. Curtiss, was devoted to a symposium on "The Maxwell field equations and the theory of relativity." Papers were read by Professors Max Mason and A. C. Lunn; the discussion was participated in by Professors E. W. Brown and Alexander Ziwet. Synopses of the symposium papers follow:

I. "Electromagnetic Field Equations," by Professor MAX MASON.

1. Definition of the vectors E , B . The field equations for charge and free space, and the retarded potentials.

2. The Lorentz electron hypothesis. The electromagnetic equations for ponderable bodies; actual and averaged value of the vectors.

3. Comparison with the action at a distance theories (Weber, Clausius et al.) and with the action in the medium theories (Maxwell, Hertz).

4. The equation of activity and of energy. The dynamics of the electron.

5. Types of analytical problems in field determinations.

6. Ponderable matter and free electrons; thermionic space charge and current.

References.

Whittaker, History of the Theories of Aether and Electricity. Encyclopädie der mathematischen Wissenschaften, V. 12 (Reiff and Sommerfeld), V. 13, 14 (Lorentz).

Lorentz, The Theory of Electrons.

II. "The Theory of Relativity," by Professor A. C. LUNN.

1. Terrestrial mechanics and geometric optics on a laboratory scale. The experimental aspect of euclidean metrics, optical meaning of straightness. The materialization of space by the ideal rigid body. Practical chronometers. The group of transformations of newtonian mechanics. The gravitational field. Large scale geodesy, with possible non-euclidean phrasing.

2. Celestial mechanics. Extension of application of the general laws of mechanics and success of the law of gravitation when observations are reduced on the basis of euclidean triangulation and rectilinear path of light. Exception in

the case of the line of apses of Mercury. Determination of inertial systems. The "absoluteness" of rotation.

3. Optics and ether theories. Development of the wave-theory of light in its successive forms. Experimental basis of the notion of fixed ether. Problem of optical determination of absolute motion and negative outcome of the Michelson-Morley experiment. The Fitz-Gerald contraction as a presumptive physical phenomenon.

4. Electromagnetic theory and electrons. Distinct physical effects of ether displacement and polarization currents. The Larmor-Lorentz equations and their group of transformations. Comparison with newtonian relativity.

5. Einstein's first or special form of relativity in terms of the constancy of the velocity of light and the mutual dependence of space and time measurements. Minkowski's theory of euclidean four-dimensional space-time and the corresponding vector analysis or algebraic covariance theory associated with the invariant sum of squares. Real interpretations of non-euclidean type. The tensor of stress, momentum and density.

6. General relativity in terms of covariant theory of general quadratic differential form in four variables. The "local" affine vector analysis of an infinitesimal region associated with the group of all linear transformations. Curvature of finite regions of space-time and application of covariant differentiation to the statement of physical relations. The Riemann-Christoffel four-index tensor and the associated reduced tensor.

7. The general dynamical tensor and Einstein's law of gravitation. Newton's law as a first approximation. Equations of motion for particles in the void. Path of a ray of light. Explanation of the residual in the case of Mercury and prediction of bending of ray of light passing near the sun. Outlook on other phenomena. Relation to molecular theories.

References.

- Einstein: *Annalen der Physik*, volume 17 (1905), pages 891-921 and volume 49 (1916), pages 769-822; *Jahrbuch der Radioaktivität*, volume 4 (1907), pages 411-462.
 Laub and Ishiwara: *Jahrbuch der Radioaktivität*, volume 7 (1910), pages 405-463 and volume 9 (1912), pages 560-648.
 Minkowski: *Göttinger Nachrichten*, 1908, pages 53-111, and *Mathematische Annalen*, volume 68 (1910), pages 472-525.

Eddington: Report on the Relativity Theory of Gravitation. London, Fleetway Press, 1918.

At the sessions of Friday and Saturday forenoons, presided over by Professor Carmichael, chairman of the section, and Professor Wilczynski respectively, the following papers were read:

(1) Professor G. A. MILLER: "Transitive constituents of the group of isomorphisms of an abelian group of order p^m ."

(2) Professor G. E. WAHLIN: "On the unessential divisors of the discriminants of a domain."

(3) Professor E. B. STOFFER: "Seminvariants of a general system of linear homogeneous differential equations."

(4) Professor E. J. WILCZYNSKI: "A set of characteristic properties of a class of congruences connected with the theory of functions."

(5) Professor A. B. COBLE: "Porisms."

(6) Professor O. E. GLENN: "Modular covariants of formal type."

(7) Professor T. E. MASON: "New pairs of amicable numbers."

(8) Professor DUNHAM JACKSON: "On functions of closest approximation."

(9) Professor R. D. CARMICHAEL: "On the expansion of certain analytic functions in series."

(10) Professor R. D. CARMICHAEL: "Boundary value and expansion problems: Algebraic basis of the theory" (preliminary communication).

(11) Professor R. D. CARMICHAEL: "Boundary value and expansion problems: Formulation of various transcendental expansion problems" (preliminary communication).

(12) Professor D. R. CURTISS: "On Jensen's extension of Rolle's theorem."

(13) Professor K. P. WILLIAMS: "The spiral descent of an airplane."

(14) Dr. GEORGE RUTLEDGE: "An invariant area property of polynomial curves of odd degree and the direct evaluation of Cotes's coefficients."

(15) Professor P. R. RIDER: "The minimum area between a curve and its caustic."

(16) Professor W. D. MACMILLAN: "On the moment of inertia in the problem of n bodies."

(17) Dr. J. R. KLINE: "Concerning the boundary of uniformly connected 'im kleinen' domains."

(18) Professor E. H. MOORE: "On the reciprocal of the general algebraic matrix."

(19) Professor W. H. ROEVER: "Determination of the wind structure from the flight of a pilot balloon."

The papers of Professors Miller, Wahlin, Stouffer, Glenn, and of Dr. Rutledge were read by title. Abstracts of the papers, numbered in accordance with the above list, are given below:

1. An abelian group cannot contain more than two characteristic operators. Hence the group of isomorphisms I of an abelian group G , when represented on letters corresponding to non-characteristic operators of G , cannot contain more than two transitive constituents which are separately simply isomorphic with I and a necessary and sufficient condition that it contains two such constituents is that the order of G is twice an odd number. When the order of G is p^m , p being a prime number, then the I of G contains only one transitive constituent which is simply isomorphic with G . Professor Miller investigates the relations existing between the various constituents of I , especially those corresponding to operators of the same order contained in G . In particular, he proves that to the identity of one such constituent there corresponds an abelian group of order p^a and of type $(1, 1, 1, \dots)$ in the next higher constituent, and he determines the orders of these groups.

2. Various investigators have made extensive studies of the unessential divisors of the discriminants of the integers of an algebraic domain. In his *Theorie der algebraischen Zahlen*, Hensel shows that if p is a common unessential divisor of the discriminants of all integers of a domain and if in the domain all the prime factors of p are of the first degree, then p is necessarily less than the degree of the domain. In the case when all the prime factors of p are not of the first degree, he finds a larger upper limit for the primes which may be common unessential divisors. In the *Mathematische Annalen*, volume 73, von Zylinski shows that p is always less than n , if p is a common unessential discriminant divisor. In the present paper, Professor Wahlin shows that in case all the prime divisors of p are of higher degree than the first, a still smaller limit can be fixed.

3. Consider the system of linear homogeneous differential equations

$$y_i^{(m)} + \sum_{l=0}^{m-1} \sum_{k=1}^n \binom{m}{l} p_{ikl} y_k^{(l)} = 0 \quad (i = 1, 2, \dots, n),$$

where p_{ikl} are functions of the independent variable x . It is known that the most general transformation of the dependent variables which leaves this system unchanged in form is given by the equations $y_k = \sum_{\lambda=1}^n \alpha_{k\lambda}(x) \bar{y}_\lambda$, $|\alpha_{k\lambda}| \neq 0$, where $\alpha_{k\lambda}$ are arbitrary functions of x . A function of the coefficients and their derivatives which has the same value for the original system as for the transformed system is called a seminvariant. Professor Stouffer calculates a complete system of seminvariants for the above set of equations. The seminvariants for the simpler case when $p_{ikm-1} = 0$, were first obtained, and the seminvariants for the general case were obtained from them by simple transformations.

4. If two complex variables, z and w , are transferred to the same sphere, and if the points which correspond to each other in a functional relation between z and w are joined by straight lines, the lines thus obtained form a congruence. The general properties of such congruences were determined by Professor Wilczynski in a paper in the *Transactions*, October, 1919. In the present paper, Professor Wilczynski shows that a certain list selected from these properties is characteristic of such congruences. It is a notable fact that most of these properties are of a purely projective character.

5. Professor Coble's paper is concerned with multiple binary forms which have the closure property. A poristic form of this sort defines a complementary poristic form such that the product of the two can be expressed as a determinant which by its very nature is poristic.

6. Professor Glenn's paper presents in brief the main results developed in the author's memoir on formal invariancy, read before the Society in December 1919 (see abstract in this BULLETIN for March, 1920). Certain further developments of general theory are included.

7. By making use of Lehmer's Table of Primes, Professor Mason has been able to show that the numbers 256·8520191

and $256 \cdot 257 \cdot 33023$ are a pair of amicable numbers. This pair of numbers belongs to one of the types treated by Euler, but the large prime was beyond the limits of his table of primes. Other number pairs are obtained by Euler's methods or modifications of his methods.

8. The determination of the best polynomial of approximation for a given continuous function $f(x)$ in a given interval, the degree of the polynomial being given, depends on the meaning attached to the phrase "best approximation." The polynomial for which the maximum of the absolute value of the error is as small as possible is known as the Tchebychef polynomial, and has been extensively studied. The polynomial for which the integral of the square of the error is a minimum is the sum of the first terms of the Legendre's series for $f(x)$, and its properties are also well known. Professor Jackson considers the polynomials for which the integral of the m th power of the error is a minimum, where m is any even positive integer, or, more generally, the integral of the m th power of the absolute value of the error, where m is any number greater than 1. It is found that certain theorems which are well known for $m = 2$ are applicable in the more general case; and it is shown that the approximating polynomial corresponding to exponent m approaches the Tchebychef polynomial as a limit when m becomes infinite. The discussion is put in such a form as to apply also to approximation by finite trigonometric sums, for example, and generally to approximation by linear combinations of a given set of linearly independent functions, having such further properties as to insure the uniqueness of the best approximating function in the sense of Tchebychef.

9. For a certain class $S(x)$ of cases of the $\bar{\Omega}$ -series introduced by Professor Carmichael in his paper in the *Transactions*, volume 17, he now treats anew the problem of the representation of functions in such series. The basic function $g(x)$ is restricted to the special asymptotic form

$$g(x) \sim x^{\mu - \rho x} e^{\alpha + \beta x} \left(1 + \frac{a_1}{x} + \frac{a_2}{x^2} + \dots \right),$$

valid in a suitable sector V . The object of the present note is to derive certain necessary and sufficient conditions for the

representation of functions in the form of series $S(x + a)$, where a is a suitable constant depending on the function to be represented.

10. Lying back of several well-known expansion problems in the theory of integral equations and of linear differential equations, both ordinary and partial, is a corresponding problem for linear algebraic equations of which the transcendental problems are limiting cases, each limiting case being realized in a way peculiar to its own class. The object of Professor Carmichael's second paper is to study systematically and to generalize in several directions the algebraic problems which thus come into the foreground, the primary purpose being to come to such understanding of these algebraic matters as will make them serve in the fullest way possible as a guide to the formulation and solution of important transcendental problems of expansion.

11. With the results of his second paper as a guide, Professor Carmichael formulates in his third paper several transcendental expansion problems (in association with certain boundary conditions when these are relevant) having to do with equations and systems of equations of the following classes: linear differential equations, both ordinary and partial; linear difference equations, both ordinary and partial; integral equations; integro-differential equations; integro-difference equations; differentio-difference equations, and some others. The systems mentioned may be made up of equations of only one class or of equations from two or more different classes. This preliminary communication announces the formulation of these problems and a certain small progress toward their solution.

12. The theorem of Jensen according to which the complex roots of the derived functions of a polynomial with real coefficients must lie within ellipses of a certain system gives little further information regarding their location. In the present paper, Professor Curtiss obtains results of a more precise nature, and applies these to the problem of locating the roots of the polynomial when the roots of a derivative are known.

13. The orientation of an airplane when making a spiral descent is determined by two angles. One of these angles is given at once in terms of the angle of attack of the machine,

and the other is obtained from the root of a cubic equation given by Devillers. Professor Williams gives a graphical solution of this cubic equation, and also the explicit expansion of the root in question by means of the formula of Lagrange.

14. Newton, in 1711, and Cotes, in 1722, used polynomial curves for approximating definite integrals. Cotes computed the coefficients which bear his name for polynomials up to degree ten. In the present paper, Dr. Rutledge evaluates, by means of determinants, Cotes's coefficients for the general case of even order. A fact brought out by this evaluation is that the area from $x = a$ to $x = b$ under any polynomial curve of degree $2n + 1$ through $2n + 1$ points equally spaced as to abscissas is an invariant of the set of curves of degree $2n + 1$ through these points.

15. Professor Rider's paper considers the problem of joining two fixed points by a curve which with its caustic and the reflected rays at the fixed points will enclose a minimum area. Euler proposed and solved a similar problem concerning a curve and its evolute. For the evolute problem the minimizing curve is a cycloid. For the caustic problem it is a transcendental curve of more complex type. Its equations are obtained in parametric form. The determination of the arbitrary constants in the solution is considered. Finally, a solution is given of a more general problem which includes the evolute and caustic problems as special cases.

16. For the problem of n bodies, Eddington has given the differential equation for the moment of inertia

$$1/2 \cdot d^2 I / dt^2 = 2T + W,$$

where T is the kinetic energy and $W = -k^2 \sum \sum m_i m_j / r_{ij}$ is the potential energy, so that $T + W = E$ is the total energy. If E is positive it is evident that I'' is always positive and ultimate dispersion is inevitable. In a steady state of motion, $2T + W = 0$, as was remarked by Poincaré. Professor MacMillan supposes that a system of bodies maintains its configuration in undergoing a process of expansion and contraction. Under this hypothesis, W is inversely proportional to the square root of I and the equation for I may be written $1/2 \cdot d^2 I / dt^2 = 2E + k_1^2 / \sqrt{I}$. The integrals of this equation show that if E is negative the period of the oscillation is

$P = k_1^2/(-2E)^{3/2}$, which is independent of the amplitude of the oscillation. If the amplitude of the oscillation be increased to the point where the system collapses ($I = 0$), then the maximum value of \sqrt{I} is twice the value which it has for the steady state in which the amplitude is zero. In other words, the system could not expand to double its size for a steady state without eventually collapsing. Applications are made to star clusters.

17. Dr. Kline shows that if R is a uniformly connected "im kleinen," bounded, two-dimensional domain whose exterior is also uniformly connected "im kleinen," then there exists a finitely connected domain Q whose boundary consists of a finite number of simple closed curves, no two of which have a point in common, such that $Q - R$ is a totally disconnected set. Using this result, he shows that necessary and sufficient conditions that a bounded two-dimensional domain R be finitely connected and bounded by a finite number of simple closed curves, no two of which have a point in common, are (1) that R be fully expanded, and (2) that the exterior of R be uniformly connected "im kleinen." The notion of a set being uniformly connected "im kleinen" on every convergent sequence belonging to the set is then defined. It is shown that any bounded set M , that is uniformly connected "im kleinen" on every convergent sequence belonging to M , is uniformly connected "im kleinen." This is not necessarily true, however, if M is unbounded. Using this notion, it is shown that if R is a proper subset of a two-dimensional euclidean space which is unbounded and simply connected, then a necessary and sufficient condition that R have as its boundary an open curve is that R be uniformly connected "im kleinen" on every convergent sequence.

18. In this paper Professor Moore calls attention to a useful extension of the classical notion of the reciprocal of a non-singular square matrix. Consider any $m \times n$ matrix κ , i.e., an array in m rows and n columns of mn complex numbers $\kappa(uv)$ ($u = 1, \dots, m; v = 1, \dots, n$). There exists one and only one $n \times m$ matrix λ , the reciprocal of κ , such that (1) the columns of λ are linear combinations of the conjugates of the rows of κ , (2) the rows of λ are linear combinations of the conjugates of the columns of κ , (3) the matrix $TS\kappa\lambda\kappa$ obtained by matricial composition of the matrices κ, λ, κ is the original

matrix κ : $TS\kappa\lambda\kappa = \kappa$, viz., $\kappa(uv) = \sum_{ts} \kappa(ut)\lambda(ts)\kappa(sv)$ (uv), where u, s run from 1 to m , and v, t run from 1 to n while S ; T denote summations over the respective ranges 1, \dots , m ; 1, \dots , n . If κ is of rank r , then λ is given explicitly as follows:

$$(r \geq 2) \lambda(v_1u_1) = \sum_{\substack{u_2 < \dots < u_r \\ v_2 < \dots < v_r}} \kappa \begin{pmatrix} u_2 & \dots & u_r \\ v_2 & \dots & v_r \end{pmatrix} \bar{\kappa} \begin{pmatrix} u_1 u_2 & \dots & u_r \\ v_1 v_2 & \dots & v_r \end{pmatrix} \\ \div \sum_{\substack{t_1 < \dots < t_r \\ t_1 < \dots < t_r}} \kappa \begin{pmatrix} t_1 & \dots & t_r \\ t_1 & \dots & t_r \end{pmatrix} \bar{\kappa} \begin{pmatrix} t_1 & \dots & t_r \\ t_1 & \dots & t_r \end{pmatrix} \quad (u_1v_1);$$

$$(r = 1) \lambda(vu) = \bar{\kappa}(uv) \div \sum_{st} \kappa(st)\bar{\kappa}(st) \quad (uv);$$

$$(r = 0) \lambda(vi) = 0 \quad (uv);$$

where as usual $\kappa \begin{pmatrix} g_1 & \dots & g_k \\ h_1 & \dots & h_k \end{pmatrix}$ denotes the determinant of the k^2 numbers $\kappa(g_i h_j)$ ($i, j = 1, \dots, k$) and $\bar{\kappa} \begin{pmatrix} g_1 & \dots & g_k \\ h_1 & \dots & h_k \end{pmatrix}$ denotes the determinant of the conjugate numbers $\kappa(g_i h_i)$.

The relation between κ and λ is mutual: κ is the reciprocal of λ , viz., (4), (5): the columns (rows) of κ are linear combinations of the conjugates of the rows (columns) of λ ; (6) $ST\lambda\kappa\lambda = \lambda$. The linear combinations of the columns of $\kappa(\lambda)$ are the linear combinations of the conjugates of the rows of $\lambda(\kappa)$ and constitute the m -ary vectors μ (the n -ary vectors ν) of a linear r -space M (N) lying in the complete m -space (n -space) of all m -ary (n -ary) vectors. Let \bar{M} (\bar{N}) denote the conjugate space of the conjugate vectors $\bar{\mu}$ ($\bar{\nu}$). Then the matrices $\kappa\lambda$ establish 1-1 linear vector correspondences between the spaces M, \bar{M} and the respective spaces N, \bar{N} ; $\mu = T\kappa\nu$ is equivalent to $\nu = S\lambda\mu$ and $\bar{\nu} = S\bar{\mu}\kappa$ is equivalent to $\bar{\mu} = T\bar{\nu}\lambda$.

19. Professor Roeber shows how, under certain assumptions, it is possible to determine from a single pilot balloon flight the nature of the vector field which represents the wind structure. For this purpose the wind chart and the differential equations of motion of the balloon are needed. These differential equations are obtained from those of a projectile by substituting for the weight of the projectile the lift of the balloon. The wind chart furnishes the finite equations of motion of the balloon. From these two sets of equations the form of the vector field along the path of the balloon is determined, and, under the given assumptions, this is sufficient to

determine the nature of the vector field representing the wind structure.

ARNOLD DRESDEN,
Secretary of the Chicago Section.

THE APRIL MEETING OF THE SAN FRANCISCO SECTION

THE thirty-fifth regular meeting of the San Francisco Section was held at Stanford University on Saturday, April 10. The chairman, Professor Blichfeldt, presided. The total attendance was twenty-three, including the following thirteen members of the Society:

Professor R. E. Allardice, Professor B. A. Bernstein, Professor H. F. Blichfeldt, Professor Thomas Buck, Professor Florian Cajori, Professor M. W. Haskell, Professor L. M. Hoskins, Professor D. N. Lehmer, Professor W. A. Manning, Professor H. C. Moreno, Dr. F. R. Morris, Professor C. A. Noble, Dr. Pauline Sperry.

The action of the executive committee in making plans for a meeting of the Section at the University of Washington June 17-19, in connection with the Pacific division of the American association for the advancement of science was approved. It was decided that this should be a special meeting of the Section, the regular Fall meeting to take place October 23, 1920, as scheduled.

The following papers were presented and discussed:

(1) Professor H. F. BLICHFELDT: "On the approximate representation of irrational numbers" (preliminary report).

(2) Professor FLORIAN CAJORI: "Note on the history of divergent series."

(3) Professor L. M. HOSKINS: "Note on the Lorentz transformation and the notion of time."

(4) Professor E. T. BELL: "The twelve elliptic functions related to sixteen doubly periodic functions of the second kind."

(5) Professor E. T. BELL: "Certain remarkable sums related to 3, 5, and 7 squares."

(6) Professor E. T. BELL: "Parametric solutions for a fundamental equation in the general theory of relativity."