NOTE ON RIEMANN SPACES.

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1. IT is proposed to establish the theorem that every closed orientable n-dimensional manifold can be represented on an n-dimensional hypersphere as a Riemann space, or generalized Riemann surface. The argument will be carried through explicitly for the case $n = 3$ only, since the extension to higher dimensions is perfectly automatic and requires scarcely more than the modification of a few subscripts.

2. We know that a 3-dimensional manifold can always be built up out of the points and boundary points of a finite number of tetrahedral regions by suitably matching together in pairs the triangular faces of the bounding tetrahedra. Let $A_1, A_2, \cdots, A_k$ be the points of the manifold which correspond to the vertices of the tetrahedra. Then it can always be so arranged that the four points A_{i1}, A_{i2}, A_{i3}, and A_{i4} corresponding to the vertices of a tetrahedron are distinct, in which case, the bounded tetrahedral region may be designated by the symbol $|A_{i1}A_{i2}A_{i3}A_{i4}|$, where the ordering of the letters A_{ij} is immaterial.

We shall say that every permutation of the letters in the symbol for a tetrahedral region determines a sense on the region and that two permutations determine the same or opposite senses according as they differ by an even or an odd number of transpositions. A tetrahedral region becomes sensed, or oriented, by association with one of the permutations and is then conveniently designated either by the symbol for the sense-giving permutation or by the symbol for any other permutation taken with a plus or minus sign according as the second permutation differs from the first by an even or an odd number of transpositions. Thus, we have

$$A_1A_2A_3A_4 = -A_2A_3A_4A_1 = A_3A_4A_1A_2 = \cdots,$$

but

$$A_1A_2A_3A_4 \neq A_2A_3A_4A_1, \text{ etc.}$$

3. Now, consider two tetrahedral regions $|AA_1A_2A_3|$ and $|A'A_1A_2A_3|$ with a face $A_1A_2A_3$ in common. If senses be

assigned to these regions, the given senses will be said to be consistent at the face $A_1A_2A_3$ if and only if we have one of the following combinations

$$(1)\qquad \begin{array}{ccccc} AA_1A_2A_3 & \text{and} & -A'A_1A_2A_3 & \text{or} \\ -AA_1A_2A_3 & \text{and} & A'A_1A_2A_3. & \end{array}$$

The manifold is said to be orientable if senses can be assigned to all the tetrahedral regions in such a way as to be consistent at every face. The property of being or not being orientable is an invariant of the manifold and does not depend on the selection of the set of tetrahedral regions.* From now on, we shall confine the discussion to orientable manifolds and shall assume that the tetrahedral regions have all been consistently sensed.

4. Now suppose that in the space of inversion (with a single point at infinity, and therefore like the hypersphere topologically) we set up a system of axes and select k finite points $P_i(x_i, y_i, z_i)$, $i=1, 2, \cdots, k$, one for each of the points A_i, and such that no four are coplanar. Then, any four of the points, such as P_1, P_2, P_3 and P_4, determine a tetrahedron which subdivides the space into two tetrahedral regions, one of which does not contain the point at infinity. We designate the latter region by any permutation $P_{i1}P_{i2}P_{i3}P_{i4}$ of the points P_1, P_2, P_3, P_4 such that

$$\Delta(P_{i1}P_{i2}P_{i3}P_{i4}) = \begin{vmatrix} x_{i1} & x_{i2} & x_{i3} & x_{i4} \\ y_{i1} & y_{i2} & y_{i3} & y_{i4} \\ z_{i1} & z_{i2} & z_{i3} & z_{i4} \\ 1 & 1 & 1 & 1 \end{vmatrix} > 0,$$

and the one which contains the point at infinity by any permutation $P_{j1}P_{j2}P_{j3}P_{j4}$ such that

$$\Delta(P_{j1}P_{j2}P_{j3}P_{j4}) < 0.$$

With this notation, the desired Riemann space is obtained at once by merely mapping each sensed region $A_{s1}A_{s2}A_{s3}A_{s4}$ of the manifold on the region $P_{s1}P_{s2}P_{s3}P_{s4}$ of the space of inversion, in such a way, of course, that the vertex A_{st} always goes over into the point P_{st} and that the images of the boundaries of various tetrahedral regions join on to one another properly along the images of their faces and edges.

* In the Poincaré terminology, a necessary and sufficient condition that a manifold be orientable is that it have no coefficient of torsion of order n.

The proof that the mapping just defined gives the desired Riemann space rests on the fact that two determinants

$$\Delta(PP_1P_2P_3) \qquad \text{and} \qquad -\Delta(P'P_1P_2P_3)$$

have the same or opposite signs according as the points P and P' lie on opposite sides of the plane $P_1P_2P_3$ or on the same side. It therefore follows that for two contiguous regions

$$(2) \qquad \begin{array}{cccc} PP_1P_2P_3 & \text{and} & -P'P_1P_2P_3 & \text{or} \\ -PP_1P_2P_3 & \text{and} & P'P_1P_2P_3 & \end{array}$$

the points of one region in the neighborhood of a point of the face $P_1P_2P_3$ are on opposite sides of the face from those of the other region. Hence, by comparison of (1) and (2), it is seen that the mapping is uniform not only in the neighborhood of points of the manifold within a tetrahedral region but also of points on a bounding face. Therefore, the only singularities in the mapping correspond to vertices A_i or edges A_iA_j. Furthermore, the mapping is everywhere r to 1 on the space of inversion, except perhaps at the points P_i or along the edges P_iP_j, because any two points, B and C, not on a line P_iP_j can be joined by a path p which does not meet a line P_iP_j. Therefore, if the point B corresponds to n points B_1, B_2, $\cdots$, B_n of the manifold, the path p corresponds to n non-intersecting paths leading from the B_i's to an equal number of points C_i.

5. In the 3-dimensional case, a Riemann space obtained by the above construction contains, in general, a network of branch lines at each of which two or more sheets coalesce. It is easy to show that, without modifying the topology of the space, the branch system may be replaced by a set of simple, non-intersecting, closed curves such that only two sheets come together at a curve. The curves may, however, be knotted and linking.

Three-dimensional Riemann spaces have been discussed by Heegaard* and Tietze,† but neither of these mathematicians seems to have been aware of their complete generality.

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* Heegaard, Inaugural Dissertation, Copenhagen, 1898. Also: *Bulletin Société math. de France*, vol. 44 (1916) p. 161.

† Tietze, *Monatshefte für Mathematik und Physik*, vol. 19 (1908), p. 1. Cf. remark on p. 104.