

$$(14) \quad \tau = \frac{\varphi(v, \mu)}{\kappa} \left(t - \frac{x'}{2} \left[\frac{1}{c' - v} - \frac{1}{c'' + v} \right] \right) + f(v, \mu).$$

Equation (7) is replaced by

$$(15) \quad G(x', t) = G \left(x', t + a \left[\frac{1}{c' - v} + \frac{1}{c'' + v} \right] \right),$$

the equation (8) by

$$(16) \quad G(x' + a) = G(x') + \frac{c\varphi a}{2\mu\kappa} \left[\frac{1}{c' - v} + \frac{1}{c'' + v} \right],$$

whence

$$(17) \quad \xi = \frac{c\varphi x'}{2\mu} \left[\frac{1}{c' - v} + \frac{1}{c'' + v} \right] + g(x, \mu).$$

The corresponding modifications for lateral and other signals are readily made.

It appears therefore that Einstein's assumption of differentiability is unnecessary, and that the needed features of "homogeneity" are already implicit in the optical and kinematic postulates as here interpreted.

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FORMULAS FOR CONSTRUCTING ABRIDGED MORTALITY TABLES FOR DECENNIAL AGES.

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MR. GEORGE KING presented in the British Registrar-General's Report for 1914 a method of constructing abridged mortality tables which consist merely of the values appearing in an ordinary mortality table but corresponding only to each quinquennial age. The computation of the values corresponding to the other ages is eliminated by the method and an enormous saving of labor is thus gained.

Such tables promise to prove very useful in investigations in human vitality because many problems which have heretofore involved too much computation to permit much investigation now become relatively easy. For this reason it seems

well worth while to derive and present the analogous formulas for constructing mortality tables for *decennial* ages.

Tables based on decennial ages require little more than half the computation necessary for tables based on quinquennial ages after the proper statistical data are collected and properly arranged,—the computation takes little over an hour even when the incidental computation is performed by hand—give practically the same information and involve much more suitable ages (i.e., ages 10, 20, 30, etc., instead of 12, 17, 22, etc.).

First, the given population statistics and the mortality statistics are each easily arranged in the age groups 5-14,* 15-24, 25-34, etc. Then, the following formula† to second differences‡ used for interpolating ordinates among areas

$$(1) \quad u_{x/t} = w_0 + (2x - t + 1) \frac{\Delta w_0}{2t^2} + \{3x^2 + 3x(1 - 2t) + (1 - 3t + 2t^2)\} \frac{\Delta^2 w_0}{6t^3},$$

where

$$w_{x/t} = u_{x/t} + u_{(x+1)/t} + \cdots + u_{(x+t-1)/t}$$

and $w_{x/t}$ is generally abbreviated to w_{nx} , is applied to determine the population and the deaths for a suitable individual age. Thus, if w_0 , w_1 , and w_2 refer to the populations (or deaths) for the age groups 5-14, 15-24, and 25-34 respectively, $u_{x/t}$ for $t = 10$ and $x = 15$ gives the population (or deaths) for the individual age 20, and

$$(2) \quad \begin{aligned} u_{15/10} &= .1(w_0 + \Delta w_0) + \frac{.01}{2} (\Delta w_0 - .3\Delta^2 w_0) \\ &= .1w_1 + \frac{.01}{2} (\Delta w_0 - .3\Delta^2 w_0) \end{aligned}$$

and similarly for ages 30, 40, etc.

Knowing the population L_x and the number of deaths d_x at an individual age x , the value of the probability of living one year or p_x is obtained by the well-known formula

* The great variation and uncertainties of the death rate at ages in the neighborhood of birth necessitate the omission of the age group 0-5.

† *Quarterly Publications of the American Statistical Association*, Dec. 1916.

‡ It is generally accepted that second differences are sufficient when dealing with ordinary statistical data.

$$(3) \quad p_x = 1 - q_x = \frac{L_x - \frac{1}{2}d_x}{L_x + \frac{1}{2}d_x}.$$

Probably, a better plan for computing values of p_x is to combine formulas (2) and (3) to give

$$p_{x+5} = \frac{.1(W_x - w_x) + \frac{.01}{2} \{(\Delta W_{x-10} - \Delta w_{x-10}) - .3(\Delta^2 W_{x-10} - \Delta^2 w_{x-10})\}}{.1(W_x + w_x) + \frac{.01}{2} \{(\Delta W_{x-10} + \Delta w_{x-10}) - .3(\Delta^2 W_{x-10} + \Delta^2 w_{x-10})\}},$$

where W_x and w_x refer to the population and *one-half* the deaths respectively, corresponding to the decennial age group running from x to $x + 9$ and the subscripts are changed to a more workable form to correspond to the ages. If we write D_x for $W_x - w_x$ and S_x for $W_x + w_x$ and notice that

$$\Delta^n W_x \pm \Delta^n w_x = \Delta^n (W_x \pm w_x) = \Delta^n S_x \quad \text{or} \quad \Delta^n D_x$$

the formula given above reduces to

$$(4) \quad p_{x+5} = \frac{D_x + \frac{.1}{2}(\Delta D_{x-10} - .3\Delta^2 D_{x-10})}{S_x + \frac{.1}{2}(\Delta S_{x-10} - .3\Delta^2 S_{x-10})}.$$

If we let $t = 10$ and $x = 5$ in formula (1), we obtain in like manner

$$(4') \quad p'_{x+5} = \frac{D_x + \frac{.1}{2}\Delta D_x - .06\frac{1}{2}\Delta^2 D_x}{S_x + \frac{.1}{2}\Delta S_x - .06\frac{1}{2}\Delta^2 S_x},$$

which may be applied to the age groups 5-14, 15-24, and 25-34 to give p_{10} .

To apply formulas (4) and (4'), columns of values of D_x and S_x are first obtained by inspection from columns of values of W_x and w_x ; the divisions are then performed by logarithms to give a column of values of $\log p_x$.

Values of $\log {}_{10}p_x$ are next obtained from values of $\log p_x$ as follows: If we sum the ten expressions obtained by sub-

stituting $10/10$, $11/10$, \dots , $19/10$ in Newton's formula (to second differences)

$$u_x = u_0 + x\Delta u_0 + \frac{x(x-1)}{2} \Delta^2 u_0,$$

we obtain

$$w_1 = 10u_0 + 14\frac{1}{2}\Delta u_0 + 3\frac{2}{3}\Delta^2 u_0,$$

which may be written

$$(5i) \quad w_1 = 10u_1 + 4\frac{1}{2}\Delta u_0 + 3\frac{2}{3}\Delta^2 u_0,$$

where, however, the true coefficient of $\Delta^2 u_0$ is 3.675. The arbitrary change in coefficients simplifies the computation considerably while the error thus introduced will generally prove insignificant in the final results.

Since $\log_{10} p_x = \log p_x + \log p_{x+1} + \dots + \log p_{x+9}$, formula (5i) enables one to obtain a column of values $\log_{10} p_{20}$, $\log_{10} p_{30}$, etc., from a column of values $\log p_{10}$, $\log p_{20}$, etc. The value of $\log_{10} p_{10}$ is obtained by using the formula

$$(5i') \quad w_0 = 10u_0 + 4\frac{1}{2}\Delta u_0 + .8\frac{1}{4}\Delta^2 u_0,$$

which is the sum of the ten expressions obtained by substituting $0/10$, $1/10$, \dots , $9/10$ in Newton's formula.

Assuming a suitable radix or number of individuals living at age 10, the values of $\log_{10} p_x$ are added accumulatively to the logarithm of the radix to give in succession the logarithm of the number of survivors at each subsequent tenth age.

The radix is usually assumed to be 100,000 at age 10 but after considerable experience with the difficulties encountered in trying to complete such mortality tables at the higher ages the writer has become convinced that the radix of at least an *abridged* mortality table should not ordinarily be greater than 1,000 at age 10. Even then in a few exceptional cases it has been found necessary to adopt rather arbitrary measures to complete the tables at the higher ages. As the values corresponding to the highest ages have very little effect on the values at the earlier ages, the adoption of a radix of 1,000 obviates the necessity of introducing arbitrary and often questionable measures *for unimportant details*.

The column of survivors for decennial ages obtained in the manner just explained constitutes the abridged mortality table desired, but a column of expectations of life is always desirable and is obtained as follows. If we sum the expressions obtained by substituting $11/10$, $12/10$, \dots , $20/10$ in Newton's formula, we get

$$(5t) \quad w_{11/10} = 10u_1 + 6\frac{1}{2}\Delta u_0 + 4\frac{2}{3}\Delta^2 u_0,$$

where the coefficient $4\frac{2}{3}$ is written instead of the true value 4.675 to simplify computation.

If formula (5t) is applied to the column of survivors 1_{10} , 1_{20} , etc., constituting the abridged mortality table, we obtain the number of survivors corresponding to the age groups 20–29, 30–39, etc. The number of survivors corresponding to the age group 10–19 is obtained by applying the formula

$$(5t') \quad w_{1/10} = 10u_0 + 5\frac{1}{2}\Delta u_0 + .8\frac{1}{4}\Delta^2 u_0,$$

which is the sum of the expressions obtained by substituting $1/10$, $2/10$, \dots , $10/10$ in Newton's formula.

By way of distinction, formulas (5i) and (5i') have been called *initial* forms and formulas (5t) and (5t') *terminal* forms, for obvious reasons.

If, now, the column of survivors for the decennial age groups be summed accumulatively, beginning at the highest ages and the successive sums be divided by the number of survivors at the individual age just preceding in each case—as given in the abridged mortality table—a column of values of *curtate* expectation of life is obtained. Assuming, however, that the average individual lives a half year in the year of his death, it is customary to add 0.5 to each of the values just mentioned to obtain values of the *complete* expectation of life—the values usually quoted as the expectation of life.

In computing the expectation of life there is no satisfactory way of determining the number of survivors corresponding to the last age group. This fact should give no serious concern, for beyond much doubt the errors in the data at the higher ages—especially those due to the consistent overstatement of ages—render worthless any special efforts to be accurate at these ages. If only one decimal place is kept in the values of the expectation of life, it will make no difference at all what the number of survivors corresponding to the last age group is taken to be, because it will not affect any of the values of the expectation of life except those at the very highest—say, two or three—decennial ages and these values because of their inaccuracy should not be given very much weight.

The values of the death rates q_x are obtained from the values of p_x by means of the relation between q_x and p_x given in formula (3).