ON THE NUMBER OF REPRESENTATIONS OF $2n$ AS A SUM OF $2r$ SQUARES.

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1. OWING possibly to its connection with X-ray analyses of crystal structure, interest in the problem of representing an integer as a sum of integral squares has recently revived. We shall first summarize briefly so much of what is known of the problem as will put the formulas established below in their proper light. Let $N(n, 2r)$ denote the total number of decompositions of n into a sum of $2r$ squares. Then, for $r = 1, 2, 3, 4$, the complete results concerning $N(n, 2r)$ are implicit in sections 40, 41, 42, 65 of the Fundamenta Nova. Jacobi, however, left the explicit statement of all but one of his results to others. When $r \ngtr 4$, $N(n, 2r)$ is expressible in terms of the divisors of n alone. By arithmetical methods, independently of elliptic functions, Eisenstein* proved some of Jacobi's results, showed how the rest might be obtained from his own theorems, and proved that, for $r > 4$ and n general, $N(n, 2r)$ can not be expressed in terms of the divisors of n alone. Letting $\xi_s'(n)$ denote the excess of the sum of the sth powers of all those divisors of n that are of the form $4k + 3$ over the like sum for all divisors of the form $4k + 1$, and $\zeta_s(n)$ the sum of the sth powers of all the divisors, Eisenstein stated a notable exception to his general theorem; showing that at least once, when n is suitably chosen, $N(n, 2r)$, for $r > 4$, may be expressed in terms of $\xi_s'(n)$, or $\zeta_s(n)$. E.g., $N(4k + 3, 10) = 12\xi_4'(4k + 3)$; or what may be shown is ultimately the same thing:† $N(8k + 6, 10) = 204\xi_4'(4k + 3)$. Liouville derived a similar result for $N(2m, 12)$ in terms of $\zeta_5(2m)$. He used for this purpose certain remarkable formulas‡ which, however, he did not prove, and which it is the

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* *Crelle*, vol. 35 (1847), p. 135.

† Either result follows from the other on applying a transformation of the second order to the theta equivalent of the appropriate Liouville formula of the kind in § 2.

‡ *Jour. des Math.* (2), vol. 6 (1861); two papers, p. 233, 369 et seq. The formulas for proper representations may be proved similarly to those in this paper.

object of this paper to establish; he also showed how the discovery of such results as Eisenstein's and his own can be made to depend upon the formulas mentioned. In 1907 G. Humbert* and K. Petr† independently proved Eisenstein's 10-square, and Liouville's 12-square results very simply by elliptic functions. At the same time J. W. L. Glaisher‡ published complete results for $N(n, 2r)$, $r = 1$ to 9, inventing the necessary functions for the cases $r = 5$ to 9. He remarked§ that the form of his results seems to indicate the non-existence for $r = 7, 8, 9$ of theorems similar to Eisenstein's or Liouville's for $r = 5, 6$. Recently L. J. Mordell|| has found and developed close connections between Glaisher's theorems and the elliptic modular invariants. Liouville's formulas seem to have been overlooked by later writers. They offer a direct method of attack upon the question of completely expressing $N(2n, 2r)$ in terms of the divisors of n alone; that is, of finding for what forms of n, r this is possible. The four formulas of Liouville are only the first cases of an infinite number of similar results which may be found as below, using higher powers than the first and second, or products, etc., of the elliptic series; and like results exist for $N(m, 2r)$ where m is odd, Eisenstein's theorem being a consequence of one of these. As Liouville remarks, there is an extensive theory in connection with such formulas. Here we shall merely prove his four, and show how the required coefficients may be found.

2. For m odd, Liouville's formulas are

$$\xi_s(m) \equiv (-1)^{(m+1)/2}\xi_s'(m);$$

$$(1) \qquad \zeta_{2r-1}(m) = \sum_{t=0}^{r-1} A_t N(2m, 4r, 4t+2), \qquad (r > 0);$$

$$(2) \qquad \xi_{2r}(m) = \sum_{t=0}^{r} B_t N(2m, 4r+2, 4t+2), \qquad (r \geqq 0);$$

and for $n = 2^{a+2}m$, m odd, $a \geqq 0$

$$(3) \qquad 2^{(2r+1)a}\zeta_{2r+1}(m) = \sum_{t=0}^{r-1} \alpha_t N(n, 4r+4, 4t+4), \qquad (r > 0);$$

* *Paris C. R.*, vol. 144, p. 874.

† *Archiv der Math.*, 1907, p. 83.

‡ (1) *Q. J. M.*, vol. 38. The results are summarized in (2): *Proc. L. M. S.* (2), vol. 5 (1907), pp. 479–90.

§ Loc. cit. (2), p. 487, §13.

|| *Q. J. M.*, vol. 48 (1917–18), Nos. 189, 190.

$$(4)\qquad 2^{2ar}\xi_{2r}(m) = \sum_{t=0}^{r-1} \beta_t N(n, 4r+2, 4t+4), \qquad (r > 0);$$

where $N(n, r, s)$ = the number of representations of n as a sum of r squares, of which the first s are odd with roots > 0, and the last $r - s$ even with roots $\gtreqless 0$. The coefficients A, B, α, β depend upon r, but not upon m or n. Liouville states that for all values of r

$$A_0 = B_0 = \alpha_0 = \beta_0 = 1;$$

$$C_{r-1} = 16^{r-1}, \qquad C_{r-t-1} = 16^{r-2t-1}C_t,$$

where C denotes either A or α; $B_r > 0$ when $r > 0$; and that recurring formulas exist whereby the successive coefficients may be calculated from the first. These assertions will be verified automatically in the proofs. We shall derive (1) from a comparison of the power series and Fourier developments of sn$u$; and (2), (3), (4) in a like manner from cn$u$, sn^2u, dn$u$ respectively.

3. The necessary series are, with $0! = 1$:

$$(5)\qquad \begin{aligned} \operatorname{sn} u &= \frac{2\pi}{kK}\Sigma \frac{q^{m/2}}{1-q^m}\sin\frac{m\pi u}{2K} \\ &= \sum_{r=1}^{\infty} S_{2r-1}(k^2)(-1)^{r-1}\frac{u^{2r-1}}{(2r-1)!}; \end{aligned}$$

$$(6)\qquad \operatorname{cn} u = \frac{2\pi}{kK}\Sigma \frac{q^{m/2}}{1+q^m}\cos\frac{m\pi u}{2K} = \sum_{r=0}^{\infty} C_{2r}(k^2)(-1)^r\frac{u^{2r}}{(2r)!};$$

$$(7)\qquad \begin{aligned} \operatorname{dn} u &= \frac{2\pi}{K}\left[\frac{1}{4} + \Sigma \frac{q^n}{1+q^{2n}}\cos\frac{n\pi u}{K}\right] \\ &= \sum_{r=0}^{\infty} D_{2r}(k^2)(-1)^r\frac{u^{2r}}{(2r)!}; \end{aligned}$$

$$(8)\qquad \begin{aligned} \operatorname{sn}^2 u &= 2\left(\frac{\pi}{kK}\right)^2\left[\Sigma \frac{q^m}{(1-q^m)^2} - \Sigma\frac{nq^n}{1-q^{2n}}\cos\frac{n\pi u}{K}\right] \\ &= \sum_{r=1}^{\infty} S_{2r}'(k^2)(-1)^{r-1}\frac{u^{2r}}{(2r)!}; \end{aligned}$$

the first sums being with respect to $n = 1, 2, 3, 4, \cdots$, and $m = 1, 3, 5, 7, \cdots$. The S, C, D, S' are polynomials in k^2 with positive integral coefficients, and their known forms are

$$(9)\qquad \begin{aligned} S_{2r-1}(k^2) &= \sum_{t=0}^{r-1} s_t k^{2t}; & C_{2r}(k^2) &= \sum_{t=0}^{r-1} c_t k^{2t};\\ S_{2r}'(k^2) &= \sum_{t=0}^{r-1} s_t' k^{2t}; & D_{2r}(k^2) &= k^{2r}C_{2r}(1/k^2), \end{aligned}$$

the last from the relation $\mathrm{cn}(ku, 1/k) = \mathrm{dn}u$. Also, for all values of r we have *

$$(10)\qquad \begin{aligned} &s_0 = c_0 = 1; \quad s_0' = s'_{r-1} = 2^{2r-1}; \quad s_{r-t-1} = s_t;\\ &s'_{r-t-1} = s_t'; \quad c_{r-1} = 2^{2r-2}. \end{aligned}$$

Henceforth, unless otherwise indicated, n represents an arbitrary integer > 0, m an arbitrary odd integer > 0, and summations are with respect to all m or all n. We shall need the following constants:

$$(11)\quad \Sigma \frac{q^{m/2}m^{2r-1}}{1-q^m} = \Sigma\, q^{m/2}\zeta_{2r-1}(m); \qquad \Sigma \frac{q^{m/2}m^{2r}}{1+q^m} = \Sigma\, q^{m/2}\xi_{2r}(m),$$

the right members coming directly from the definitions of ζ, ξ' in § 1, on expanding each term on the left and rearranging all in ascending powers of q. Writing for the moment

$$(12)\quad n = 2^a m, \qquad \xi_r''(n) = 2^{ar}\xi_r(m), \qquad \zeta_r'(n) = 2^{ar}\zeta_r(m),$$

we have also

$$(13)\quad \Sigma \frac{q^n n^{2r}}{1+q^{2n}} = \Sigma\, q^n \xi_{2r}''(n); \qquad \Sigma \frac{q^n n^{2r+1}}{1-q^{2n}} = \Sigma\, q^n \zeta'_{2r+1}(n).$$

The necessary theta constants are, $\vartheta_a \equiv \vartheta_a(q)$:

$$(14)\quad \sqrt{\frac{2K}{\pi}} = \vartheta_3 = \sum_{-\infty}^{+\infty} q^{n^2}; \qquad k = \frac{\vartheta_2^2}{\vartheta_3^2}; \qquad \vartheta_2 = 2\,\Sigma\, q^{m^2/4};$$

whence, for b, c integers $\geqq 0$, we have, obviously

$$(15)\qquad \vartheta_2^b(q^4)\vartheta_3^c(q^4) = \Sigma\, q^n[2^b N(n, b+c, b)].$$

For passing from representations to compositions, the following is useful:

$$(16)\qquad \vartheta_2^b(q^4)\vartheta_3^c(q^4) = \Sigma\, q^n[N'(n, b+c, b)],$$

* The third of the relations (10) is from $\mathrm{sn}(kx, 1/k) = k\,\mathrm{sn}(x, k)$; and the fourth follows from this by actually forming the product $\mathrm{sn}u \times \mathrm{sn}u$, and noting that the coefficient is necessarily a reciprocal polynomial in k^2. It is important for the verification of Liouville's α_t coefficients to observe that $s_0' = s'_{r-1} = 2^{2r-1}$, which may be seen at once from $S_{2r}'(k^2)$, whose absolute term is $\frac{1}{2}[(1+1)^{2r} - (1-1)^{2r}]$.

where $N'(n, p, q)$ is the number of representations of n as a sum of p squares, the first q of which are odd with roots $\gtrless 0$, and the last $p - q$ even with roots $\gtreqless 0$.

4. Equating coefficients of $(-1)^{r-1}u^{2r-1}/(2r-1)!$ in (5), we find, on using (11),

$$(17)\qquad \frac{4}{k}\left(\frac{\pi}{2K}\right)^{2r} \Sigma\, q^{m/2}\zeta_{2r-1}(m) = S_{2r-1}(k^2) \equiv \sum_{t=0}^{r-1} s_t k^{2t},$$

which, on substituting for k, K their values from (14), becomes, after some obvious reductions,

$$(18)\qquad 4\,\Sigma\, q^{m/2}\zeta_{2r-1}(m) = \sum_{t=0}^{r-1} s_t \vartheta_2^{4t+2}\vartheta_3^{4r-4t-2}.$$

From (6), in the same way,

$$(19)\qquad 4\,\Sigma\, q^{m/2}\xi_{2r}(m) = \sum_{t=0}^{r-1} c_t \vartheta_2^{4t+2}\vartheta_3^{4r-4t}.$$

Changing q into q^4 in (18), (19), and applying (15), we get

$$(20)\qquad \zeta_{2r-1}(m) = \sum_{t=0}^{r-1} 2^{4t} s_t N(2m, 4r, 4t+2);$$

$$(21)\qquad \xi_{2r}(m) = \sum_{t=0}^{r-1} 2^{4t} c_t N(2m, 4r+2, 4t+2);$$

by equating the coefficients of q^{2m} in the respective series. To verify Liouville's statements about the coefficients A_t, B_t of § 2, we note that (20), (21) become respectively (1), (2) on putting $A_t = 2^{4t}s_t$, $B_t = 2^{4t}c_t$; hence by (10) the verification is complete. The calculation for (8) is similar to the foregoing. We find first, on using (13),

$$\frac{2}{k^2}\left(\frac{\pi}{K}\right)^{2r+2} \Sigma\, q^n \zeta'_{2r+1}(n) = \sum_{t=0}^{r-1} s_t' k^{2t};$$

which reduces to

$$(22)\qquad 2^{2r+3}\,\Sigma\, q^n \zeta'_{2r+1}(n) = \sum_{t=0}^{r-1} s_t' \vartheta_2^{4t+4}\vartheta_3^{4r-4t};$$

and this, on expressing ζ' in terms of ζ, by (12), substituting q^4 for q, and using (16), becomes, for $n = 2^a m$,

$$(23)\qquad 2^{a(2r+1)}\zeta_{2r+1}(m) = \sum_{t=0}^{r-1} 2^{4t-2r+1} s_t' N(4n, 4r+4, 4t+4).$$

Corresponding to (22), (7) gives, on substituting for D_{2r} its value in terms of C_{2r} from (9),

$$(24)\qquad 2\left(\frac{\pi}{K}\right)^{2r+1}\Sigma\, q^n \xi_{2r}''(n) = k^{2r}\sum_{t=0}^{r-1} c_t k^{-2t};$$

whence

$$(25)\qquad 2^{2r+2}\,\Sigma\, q^n \xi_{2r}''(n) = \sum_{t=0}^{r-1} c_t \vartheta_2^{4r-4t}\vartheta_3^{4t+2};$$

$$(26)\qquad 2^{2ar}\xi_{2r}(m) = \sum_{t=0}^{r-1} c_t 2^{2r-4t-2} N(4n,\, 4r+2,\, 4r-4t),$$

the last by changing q into q^4 in (25) and equating coefficients of q^{4n}; $n = 2^a m$, $a \geqq 0$. This may clearly be rewritten

$$(27)\qquad 2^{2ar}\xi_{2r}(m) = \sum_{t=0}^{r-1} 2^{4t-2r+2} c_{r-t-1} N(4n,\, 4r+2,\, 4t+4).$$

Again, on writing $\alpha_t = 2^{4t-2r+1}s_t'$, $\beta_t = 2^{4t-2r+2}c_{r-t-1}$, we see by (10) that these coefficients have the properties announced by Liouville, and that (23), (27) are identical with (3), (4). The r-limitations $r \geqq 0$, etc., in (1) to (4) are obviously met.

5. The coefficients s, s' of (20), (23) may be calculated by recurrence, as shown by Gudermann.* The coefficients of (21), (27) are more easily found by Hermite's method.† But whichever method is used, the computation involves great labor if $r > 20$; and neither gives the c, s, s' as explicit functions of r, t. It would be desirable, for the consideration of several questions related to compositions as sums of squares, to have these functions of r, t explicitly. Thus, e.g., were they known, it would be possible, following Liouville's indications,‡ to state necessary and sufficient conditions that

* *Crelle*, vol. 19 (1839), pp. 79–83. Some of the coefficients in Gudermann's § 115 seem to be incorrect, but the essential part of his method on p. 79 is unaffected by this. Another recurrence for the s-coefficients is given by $\operatorname{sn}'u = \operatorname{cn}u\operatorname{dn}u$, and this one enables us to apply Hermite's method to the calculation of the s, s' as well as the c.

† "Rémarques sur le développement de cos amx;" *J. des Math.* (2), vol. 9 (1869), p. 289; *Paris C. R.*, vol. 77 (1863), p. 613. Apparently the quantities first denoted by $A_0, A_1, \cdots, A_n$ in Hermite's paper should each be multiplied by $(-1)^{n+1}$. In the rest of the paper they are taken with this meaning, so that the final results appear as intended. The misprint is of no importance for Hermite's purpose, but it is for ours; since otherwise the coefficients would not all be positive.

‡ *J. des Math.* (2), vol. 6, pp. 234–5. If (28)–(31) of the present paper are used in this connection, the powers of 2 do not appear as factors (as in Liouville's illustration), but only the binomial coefficients.

$N(2n, 2r)$ shall be a ξ, ζ-function of the divisors of n alone, for general r and n. The coefficients s, s', c, and hence A, B, α, β are clearly all integers > 0; but beyond this, and the few facts concerning them which were observed by Liouville, little if anything of arithmetical importance is known about them. The comparative largeness of their prime factors is significant.

6. It will be worth while, for its bearing on the question of composition, to write down the formulas which correspond to (1)–(4), when the restriction that the odd squares shall have positive roots is removed. They are, from (15), (16), (20), (21), (23), (27), as follows:

$$(28) \qquad 4\zeta_{2r-1}(m) = \sum_{t=0}^{r-1} s_t N'(2m, 4r, 4t+2),$$

$$(29) \qquad 4\xi_{2r}(m) = \sum_{t=0}^{r-1} c_t \, N'(2m, 4r+2, 4t+2),$$

$$(30) \qquad 2^{a(2r+1)+2r+3}\zeta_{2r-1}(m) = \sum_{t=0}^{r-1} s_t' \, N'(n, 4r+4, 4t+4),$$

$$(31) \qquad 2^{2(ar+r+1)}\xi_{2r}(m) = \sum_{t=0}^{r-1} c_{r-t-1} \, N'(n, 4r+2, 4t+4),$$

where m, n, r are as in § 2.

7. For $N(n, 2r)$ as defined in § 1, we have

$$N(n, 2r) = N'(n, 2r, 0) + \binom{2r}{1}N'(n, 2r, 1) + \binom{2r}{2}N'(n, 2r, 2) + \cdots + N'(n, 2r, 2r),$$

where $\binom{a}{b} = a!/b!(a-b)!$. Hence, it is easily seen from (28), a sufficient condition that $N(2m, 4r)$ shall be expressible in terms of the divisors of $2m$ alone is that the ratio $s_t : \binom{4r}{4t+2}$ shall have the same value for $t = 0, 1, 2, \cdots, r-1$. This condition is satisfied for $r = 3$. When $r = 4, 5, 6$, it is not. Or again, if some but not all of $s_1, s_2, \cdots, s_{r-2}$ are divisible by one prime $> 4r$, then clearly the condition is not satisfied. This criterion rejects $r = 5$, the prime factor 307 occurring in two out of the possible three terms; or this case is rejected by 83, which is a factor of only the middle term. Similar conclusions may be read from (29), (30), (31).

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