THE DECEMBER MEETING OF THE SAN FRANCISCO SECTION.

The thirty-second regular meeting of the San Francisco Section, postponed from October 26, was held at the University of California on Saturday, December 14, 1918. The total attendance was thirteen, including the following nine members of the Society:

Professor B. A. Bernstein, Professor Florian Cajori, Professor M. W. Haskell, Professor Frank Irwin, Professor D. N. Lehmer, Professor W. A. Manning, Dr. F. R. Morris, Professor C. A. Noble, and Dr. Pauline Sperry.

The chairman, Professor Manning, opened the meeting, later giving way to Professor Cajori, the newly elected chairman. At the election of officers the following were chosen for one year: Chairman, Professor Florian Cajori; Secretary, Professor B. A. Bernstein; Programme committee, Professor L. M. Hoskins, Professor W. A. Manning, and Professor B. A. Bernstein.

It was decided to hold the next fall meeting on Saturday, October 25, 1919, at the University of California.

The following papers were read:

(1) Professor Florian Cajori: "A note on the history of Playfair's axiom."

(2) Professor Florian Cajori: "On the Aristotelian tract, De lineis insecabilibus."

(3) Professor M. W. Haskell: "Triangles inscribed and circumscribed to a plane cubic."

(4) Professor D. N. Lehmer: "A non-tentative method of solving the indeterminate equation $Ax + By + Cz + \cdots = n$."

(5) Professor E. T. Bell: "A partial isomorph of trigonometry."

(6) Professor E. T. Bell: "Arithmetical paraphrases (I)."

(7) Professor E. T. Bell: "Arithmetical paraphrases (II); elliptic and theta series."

(8) Professor E. T. Bell: "Arithmetical paraphrases (III); chiefly for the doubly periodic functions of the first, second, and third kinds."

(9) Professor E. T. Bell: Arithmetical paraphrases (IV); class number formulas."

Professor Bell's papers were read by title. Abstracts of the papers follow below.

1. Professor Cajori points out that Playfair's axiom on parallel lines, now usually attributed to William Ludlam (1785), was used by Joseph Fenn in Dublin, in 1769.

2. The De lineis insecabilibus enumerates 5 arguments, current in Aristotle's day, in favor of the existence of indivisible lines, 22 considerations showing the invalidity of those arguments, and 26 considerations tending to disprove the view that a line is made up of points. Some of the arguments are rigorous. In Professor Cajori's opinion the tract deserves a place in the history of mathematics.

3. Professor Haskell shows that there are twenty-four triangles inscribed and circumscribed to a non-singular cubic. These twenty-four triangles occur in four sets of six each, corresponding to the four inflexional triangles, so that in each set three are triply in perspective with the other three, the centers of perspective for each pair being three collinear inflexions. The vertices of each pair lie on a conic.

For a cubic with a double point, the total number of triangles is two, which are real if the double point is isolated and imaginary otherwise. A cuspidal cubic has no such triangle.

Further, with respect to each inflexional line there is a six-point involution on the cubic, determined by a pencil of conics, where each set of six points are the vertices of two triangles triply in perspective with respect to the inflexions lying on the given line. These two triangles become coincident in three cases, the vertices then being sextactic points.

4. The solution of the indeterminate equation $Ax + By = n$ by means of the regular continued fraction representing A/B furnishes an example of the very few non-tentative methods available in the theory of numbers. Professor Lehmer uses the theory of continued fractions of higher orders to obtain all the solutions of the equation $Ax + By + Cz + \cdots = n$.

The paper will be offered to the *American Journal* as part of a memoir on continued fractions of higher orders, an abstract of which has already been inserted in the *Proceedings of the National Academy of Sciences*.

5. Professor Bell defines a function to be even or odd in a set of variables according as it does not or does change sign when the signs of all the variables are changed simultaneously. Let X_i, Y_j denote sets containing a_i, b_j variables respectively; then

$$f(X_1, X_2, \cdots, X_r | Y_1, \cdots, Y_s)$$

is defined to be even in each X, odd in each Y, and its parity is the symbol

$$p(a_1, a_2, \cdots, a_r | b_1, b_2, \cdots, b_s)$$

whose significance is obvious. If

$$a_i = 1 = b_j, \ (i = 1, \cdots, r; j = 1, \cdots, s),$$

the parity is written

$$p(1^r | 1^s).$$

It is shown that any function whose parity is

$$p(a_1, a_2, \cdots, a_r | b_1, b_2, \cdots, b_s)$$

is linearly expressible in terms of 2^κ properly chosen functions F, whose parities are all of the form

$$p(1^\alpha | 1^\beta),$$

where

$$\alpha + \beta = \kappa,$$

and

$$\kappa = (a_1 + a_2 + \cdots + a_r) + (b_1 + b_2 + \cdots + b_s) - (r + s).$$

An arbitrary function (one neither odd nor even in any set of its variables) of k variables is similarly expressible, the number of the F being 2^k. These results are useful in the arithmetical applications of the elliptic and theta functions.

6. From 1858 to 1865 Liouville published intermittently a series of eighteen memoirs, "Sur quelques formules générales qui peuvent être utiles dans la théorie des nombres," wherein he gave without proof many remarkable theorems which he and others applied to various questions of interest, including class number formulas. Proofs by extensions of the method used by Dirichlet in proving Jacobi's four square theorem have been given for most of Liouville's general results by H. J. S. Smith (1865), T. Pepin (1889–90), G. B. Mathews

(1892), and E. Meissner (1907). But, as remarked by Bachmann (1910): "Eine zusammenhängende systematische Herleitung und Verbindung dieser Formeln (Liouville's), welche ihren Quell und die Prinzipien klarlegte, nach denen die in den Formeln auftretenden Argumente der Funktionen, auf welche sie sich beziehen, zu wählen sind, würde sehr wertvoll sein." By means of the theorems in his first paper and another simple principle, Professor Bell shows that the classical expansions in the theory of elliptic and theta functions may be paraphrased directly into theorems of the Liouville kind; moreover that a simple method is thereby provided for the discovery at will of new results of the same general sort. Liouville's formulas are nearly all paraphrases of Jacobi's expansions in the Fundamenta Nova and in the memoirs on rotation.

7. In his second paper on paraphrases Professor Bell prepares and collects elliptic and theta expansions in a form suitable for paraphrase, and gives sixteen trigonometric identities basic for Liouville's numerous (unproved) theorems on the quadratic forms of certain primes. The series are classified according to the forms of the divisors d, δ appearing in the sine, cosine coefficients of the powers of q. Similar series for sets of non-doubly periodic theta quotients, such as Biehler's, Hermite's, and Humbert's, are also given.

8. In his third paper on paraphrases, Professor Bell applies the methods of his first to a few of the series in the second. Among other results it is shown that all the formulas in Liouville's fifth memoir (and many more) are immediate paraphrases of a few classic formulas relating to the doubly periodic functions of the second kind. Illustrative of the simplicity of the method, the formulas of Liouville's sixth memoir follow at once from Jacobi's series for $\text{sn}^3 u$.

9. Professor Bell's fourth paper on paraphrases applies the method to the series for non-doubly periodic theta quotients given by Hermite (1862) and extended by G. Humbert (1907). The resulting paraphrases involve class numbers and arbitrary functions odd or even in one or more variables. It is shown how such paraphrases are related to the number of representations of an integer as a sum of 3, 5, 7, 9, 11, $\cdots$ squares.

B. A. BERNSTEIN,
Secretary of the Section.