A THEOREM ON THE VARIATION OF A FUNCTION.

BY DR. PAUL R. RIDER.

THE following is a well known theorem of differential geometry:

The differential quotient $d\phi/ds$ (ds is the element of arc) of a function $\phi(u, v)$ at a point on a surface varies in value with the direction from the point. It equals zero in the direction tangent to the curve $\phi = c$, and attains its greatest absolute value in the direction normal to this curve.*

This theorem admits of a generalization if we use a more comprehensive definition of length, a definition sometimes employed in the calculus of variations. Let then

$$S = \int_{t_0}^{t_1} F(x, y, x', y')dt$$

be the generalized length of arc along a curve

$$(C) \qquad x = x(t), \quad y = y(t).$$

By reason of homogeneity conditions†

$$S = \int_{t_0}^{t_1} F(x, y, \cos\theta, \sin\theta)\sqrt{x'^2 + y'^2}dt$$
$$= \int_{s_0}^{s_1} F(x, y, \cos\theta, \sin\theta)ds,$$

in which

$$\cos\theta = \frac{x'}{\sqrt{x'^2 + y'^2}}, \quad \sin\theta = \frac{y'}{\sqrt{x'^2 + y'^2}}.$$

Then

$$A = \left|\frac{d\phi}{dS}\right| = e\frac{\phi_x dx + \phi_y dy}{F(x, y, \cos\theta, \sin\theta)ds}$$
$$= e\frac{\phi_x \cos\theta + \phi_y \sin\theta}{F(x, y, \cos\theta, \sin\theta)},$$

* See Eisenhart, Differential Geometry, pp. 82–83.

† See Bolza, Vorlesungen über Variationsrechnung, p. 194.

where subscripts indicate partial differentiation, and where e is chosen equal to ± 1 so as to make A positive. Differentiating A with respect to θ, and setting the result equal to zero, we get

$$(1)\quad F(-\phi_x \sin\theta + \phi_y \cos\theta) - (\phi_x \cos\theta + \phi_y \sin\theta)(-F_{x'} \sin\theta + F_{y'} \cos\theta) = 0,$$

$F_{x'}$, $F_{y'}$ denoting partial derivatives of F with respect to its third and fourth arguments respectively. Since

$$F = F_{x'} \cos\theta + F_{y'} \sin\theta,*$$

equation (1) reduces to

$$(2)\quad \phi_y(x, y)F_{x'}(x, y, \cos\theta, \sin\theta) - \phi_x(x, y)F_{y'}(x, y, \cos\theta, \sin\theta) = 0,$$

and if we define direction on the curve $\phi = c$ by means of the angle $\bar{\theta} = \arctan(-\phi_x/\phi_y)$, (2) becomes

$$F_{x'}(x, y, \cos\theta, \sin\theta) \cos\bar{\theta} - F_{y'}(x, y, \cos\theta, \sin\theta) \sin\bar{\theta} = 0.$$

But this equation determines the value of θ to which the curve $\phi = c$ is transversal.†

Therefore the differential quotient $d\phi/dS$ is equal to zero in the direction tangent to the curve $\phi = c$ and has its maximum absolute value in the direction to which the curve $\phi = c$ is transversal.

WASHINGTON UNIVERSITY,
ST. LOUIS, MO.

TANGENTIAL INTERPOLATION OF ORDINATES AMONG AREAS.

BY DR. C. H. FORSYTH.

(Read before the American Mathematical Society December 27, 1917.)

IF we wish to interpolate several values in each interval between the successive ordinates $u_0, u_1, u_2, \cdots, u_n$ by finite differences, only a low order of differences can with propriety be used, since high orders based on ordinary statistical data

* See Bolza, loc. cit., p. 196.
† See Bolza, loc. cit., p. 303.