6. In Professor Bell's second paper the series are for the squares and certain products of the complete set of eighteen doubly periodic theta quotients considered by Hermite in his memoir on Kronecker's class-number relations. The series have important arithmetical consequences. The paper has appeared in the *Messenger of Mathematics.*

7. While Salmon discusses the satellite line of the cubic at some length he fails to give its equation. The symbolical expression for this important covariant has been supplied by Morley. In the present paper Professor Winger derives the explicit equation of the satellite, both for the rational and general cubic, in canonical forms, and discusses associated loci. Several chain theorems are obtained and a generalization is made for the plane curve of order n.

8. Professor Winger's second paper, a preliminary report of which was made to the Society at a former meeting, is intended as a sequel to two papers of similar titles which have already appeared in the *American Journal of Mathematics.* The varieties of self-projective rational septimics in parametric form are exhibited and the more striking properties of certain of the curves are pointed out. According to the criterion of classification adopted, there are 26 distinct types, invariant under cyclic groups of order 2, 3, 4, 5, 6, and 7, dihedral groups of order 6, 10, and 14, and an infinite group.

B. A. BERNSTEIN,
Secretary of the Section.

THE APRIL MEETING OF THE AMERICAN MATHEMATICAL SOCIETY AT CHICAGO.

THE tenth regular meeting of the American Mathematical Society at Chicago, being also the forty-first regular meeting of the Chicago Section, was held on Friday and Saturday, April 12 and 13, at the University of Chicago. The various sessions were attended by about fifty persons, among whom were the following thirty-five members of the Society:

Professor R. P. Baker, Professor G. A. Bliss, Professor J. W. Bradshaw, Professor R. D. Carmichael, Professor A. R.

Crathorne, Mr. G. H. Cresse, Professor L. E. Dickson, Professor Arnold Dresden, Professor J. W. Glover, Professor E. R. Hedrick, Professor T. H. Hildebrandt, Professor A. M. Kenyon, Professor S. Lefschetz, Professor A. C. Lunn, Professor Malcolm McNeill, Professor W. D. MacMillan, Professor C. N. Moore, Professor E. H. Moore, Professor R. E. Moritz, Professor E. J. Moulton, Professor W. H. Roever, Miss I. M. Schottenfels, Dr. A. R. Schweitzer, Professor C. H. Sisam, Professor E. B. Skinner, Professor H. E. Slaught, Dr. G. W. Smith, Mr. L. L. Steimley, Professor E. B. Van Vleck, Professor L. G. Weld, Professor E. J. Wilczynski, Dr. C. E. Wilder, Professor R. E. Wilson, Professor A. E. Young, Professor J. W. A. Young.

Forty-two persons joined in a dinner at the Quadrangle Club on Friday evening, which was followed by a number of short informal speeches, President Dickson presiding.

At the session of Friday forenoon, Professor Hedrick read a paper in memory of Professor E. W. Davis, whose death occurred on February 3, 1918. A motion to transmit a copy of Professor Hedrick's paper to the family of Professor Davis was made by Professor Slaught and concurred in by a rising vote of the meeting.

Friday afternoon was devoted to a symposium on "Divergent series and summability." The principal papers were presented by Professor R. D. Carmichael, who spoke on "General aspects of the theory of summable series," and by Professor C. N. Moore, whose subject was "Applications of the theory of summability to developments in orthogonal functions." These papers were followed by a general discussion by Professors E. H. Moore, Hedrick, Curtiss, Hildebrandt, Bliss, and others.

Professor Bliss, chairman of the section, presided at the meetings on Friday; Professor Weld presided on Saturday morning.

At the sessions of Friday forenoon and Saturday forenoon, the following papers were read:

(1) Professor C. N. Moore: "On the summability of the developments in Bessel's functions."

(2) Mr. M. G. Smith: "On the zeros of functions defined by linear homogeneous differential equations, containing a parameter."

(3) Professor C. H. Sisam: "On surfaces containing a system of cubics which do not constitute a pencil."

(4) Mr. E. P. LANE: "Conjugate systems with indeterminate axis curves."

(5) Mr. R. F. BORDEN: "On the Laplace-Poisson mixed equation."

(6) Professor R. D. CARMICHAEL: "On a general class of integrals of the form $\int_0^{\infty} \varphi(t)g(x + t)/g(x)dt$."

(7) Professor S. LEFSCHETZ: "Abelian manifolds."

(8) Dr. P. R. RIDER: "On the f_1 function of the calculus of variations."

(9) Professor E. J. WILCZYNSKI: "Some geometric aspects of the theory of functions."

(10) Mr. L. J. ROUSE: "A contribution to the question of linear dependence in linear integral equations."

(11) Professor G. A. MILLER: "Determinant groups."

(12) Professor G. A. MILLER: "Group theory proof of two elementary theorems in number theory."

(13) Professor LOUIS BRAND: "Flexural deflections and statically indeterminate beams."

(14) Professor ARNOLD EMCH: "Proof of Pohlke's theorem and its generalization by affinity."

(15) Dr. A. R. SCHWEITZER: "On the iterative properties of an abstract group. Second paper."

(16) Dr. A. R. SCHWEITZER: "Concerning independence."

(17) Professor T. H. HILDEBRANDT: "On a generalization of a theorem of Toeplitz."

Mr. Smith and Mr. Borden were introduced by Professor Carmichael, Mr. Lane by Professor Wilczynski, and Mr. Rouse by Professor Hildebrandt. The papers of Dr. Rider, Mr. Rouse, Professor Miller, Professor Brand and Professor Emch were read by title.

Abstracts of the papers follow, the numbers corresponding to those in the list above.

1. The principal object of Professor Moore's paper is to extend certain results of a paper presented to the Society at the last summer meeting (see this BULLETIN, volume 24, page 65). In the previous paper the summability $(C, k > \frac{1}{2})$ at the origin, and the uniform summability $(C, k > \frac{1}{2})$ in the neighborhood of the origin, of the Bessel's development of an arbitrary function satisfying certain conditions, were established. In the present paper it is further

shown that the development is summable (C, $\frac{1}{2}$) at the origin and uniformly summable (C, $\frac{1}{2}$) in its neighborhood, under the same conditions on the arbitrary function. It is also shown that this is all that can be established for functions of the type considered, as an example is given of such a function for which the development is non-summable (C, $0 \leqq k < \frac{1}{2}$) at the origin.

2. Using a certain generalization of a theorem due to Professor R. D. Carmichael and published in the *Annals of Mathematics* for March, 1918, Mr. Smith has derived conditions under which one can be assured that any solution of a differential equation of general order n has at least a given number of zeros in a fixed (finite) interval. It is found that the theorems are most easily applied in the case of an equation of even order of the form

$$E(u) \doteq \frac{d^{2m}}{dx^{2m}}[e_{2m}(x, \lambda)u] + \lambda^2 \frac{d^{2m-2}}{dx^{2m-2}}[e_{2m-2}(x, \lambda)u] + \cdots + \lambda^{2m} e_0(x, \lambda)u = 0.$$

For the particular case of $2m = 4$, one of the theorems takes the following form: If the parameter λ in the equation $E(u) = 0$ is fixed and satisfies the relations $\lambda \geqq S$, $\lambda \geqq s\pi/(b - a)$, and if for every x in the interval $a \leqq x \leqq b$ we have

$$e_0(x, \lambda) \geqq 3e_2 - 21e_4 + (60e_4 - 6e_2)\tan^2 \lambda\,(x - a + \pi/2\lambda),$$

the equality sign not holding throughout (a, b), then any solution $u(x, \lambda)$ of $E(u) = 0$ has at least one zero in each subinterval of (a, b) of length π/λ measuring from the point a, and hence has at least s zeros in (a, b).

3. Professor Sisam's paper classifies completely the types of algebraic surfaces which contain an algebraic system of ∞^1 cubics which do not constitute a pencil, and discusses some fundamental properties of these surfaces.

4. The differential equation of the axis curves of a given conjugate system on a surface, and the conditions for their indeterminateness, were given by Professor Wilczynski in a paper

which appeared in the *Transactions*, volume 16. Making use of these conditions, Mr. Lane considers the problem of finding those surfaces on which axis curves are indeterminate. He reduces the problem to the integration of two ordinary linear homogeneous differential equations of the third order, and obtains a parametric representation for the coordinates of an arbitrary point on the most general surface of this kind. When two cones with a common vertex are given, a surface of the kind under consideration is determined, except for a three-parameter group of projective transformations, which leave invariant the common vertex of the two cones and every straight line through this point. The fundamental conjugate system on the surface then consists of the two one-parameter families of plane curves cut out on the surface by the tangent planes of the two cones. A number of geometric theorems concerning certain closely allied curves and surfaces are obtained.

5. This paper by Mr. Borden deals with the elementary theory of the mixed equation

$$(1) \quad f'(x+1) + p(x)f'(x) + q(x)f(x+1) + m(x)f(x) = 0$$

along lines initiated by Poisson (*Journal de l'Ecole Polytechnique*, tome 6 (1806), pages 127–141), and in intimate relation to the analogous theory of the Laplace equation

$$s + ap + bq + cz = 0$$

as developed for instance in Forsyth's "Theory of Differential Equations," volume 6, pages 44–96.

The mixed equation has two fundamental invariants under the group of transformations

$$f(x) = v(x)g(x).$$

If these invariants are properly chosen, the equation can be solved in finite form if one of them is zero. The equation is unchanged in form by the Laplace-Poisson transformations

$$(S) \qquad f_{S_1}(x) = f(x+1) + p(x)f(x),$$

$$(T) \qquad f_{T_1}(x) = f'(x) + q(x-1)f(x),$$

so that if the equation resulting from one or more applications of (S) or (T) to (1) has a zero invariant it may be solved in finite form. The solution of (1) is then obtained by a recurrence relation. The necessary and sufficient conditions that the above zero invariant exists are obtained.

More general transformations analogous to the one used by Lévy (*Journal de l'Ecole Polytechnique*, tome 38 (1886), page 67) are investigated and are found not to be generally useful in obtaining solutions.

6. The general class of integrals

$$J(x) = \int_0^\infty \varphi(t) \frac{g(x+t)}{g(x)} dt,$$

in which

$$g(x) \sim x^{P(x)} e^{Q(x)} \left(1 + \frac{a_1}{x} + \frac{a_2}{x^2} + \cdots \right),$$

includes several instances each of which is central in integral transformations of fundamental importance. Professor Carmichael points out that the fundamental convergence properties of these integrals are of simple character when $\varphi(t)$ and $g(x)$ are subject to certain broad general restrictions. Under these restrictions he develops the basic general elements of this convergence theory.

7. In this note Professor Lefschetz gives an account of a study of abelian manifolds from the point of view of their connectivity and their multiple integrals.

The results arrived at are closely related to some general theorems recently published by him along that line and also to the classical investigations on hyperelliptic surfaces of Picard, Humbert, and the geometers of the Italian school.

8. In a paper presented to the Society at the February meeting Dr. Rider developed a theory of the calculus of variations for n dimensions, using an integral of the form

$$\int_{t_0}^{t_1} f(x_1, \cdots, x_n, \tau_1, \cdots, \tau_{n-1}) \sqrt{x_1'^2 + \cdots + x_n'^2} dt.$$

The present paper gives the f_1-function for such an integral. It will be published in *Washington University Studies*.

9. Professor Wilczynski's paper involves a discussion of the various rational osculants determined by a given analytic function, the geometric relations between their zeros and poles, and a number of new notions closely related to intrinsic geometry.

10. In this note Mr. Rouse obtains a number of equivalent sets of conditions for linear dependence relative to linear integral forms of the type

$$\varphi_1(x) - \int_a^b k_1(x, y)\varphi_1(y)dy - \int_a^b k_2(x, y)\varphi_2(y)dy \equiv L(x)$$

which are analogous to the conditions for linear dependence which one finds in connection with the theory of linear algebraic equations. These conditions can easily be extended to the case of a system of m linear integral expressions in n functions.

11. It is known that the determinant D of order n whose n^2 elements are independent variables is invariant under a group G of order $(n!)^2$, and hence this polynomial assumes $n^2!/(n!)^2$ values when it is transformed by the substitutions of the symmetric group of degree n^2. These values may be arranged in pairs such that each pair is composed of polynomials which differ only with respect to sign and hence the square of D is transformed into itself by a substitution group K of order $2(n!)^2$. The main object of Professor Miller's paper is to study the groups G and K.

The groups G and K are simply isomorphic with imprimitive groups of degree $2n$ corresponding to the permutation of the rows and columns of D. When $n > 3$, G is a simply transitive primitive group and the subgroup composed of all its substitutions which omit a given element has two transitive constituents. The group $\underline{G}$ corresponding to the permutations of the positive terms of D under G is also a simply transitive primitive group but its subgroup composed of all the substitutions which omit a given element has a number of transitive constituents which increases indefinitely with n. The factors of composition of G are 2, 2, $n!/2$, $n!/2$, and hence the factors of composition of K are 2, 2, 2, $n!/2$, $n!/2$ and both G and K are unsolvable whenever $n > 3$.

12. The object of Professor Miller's second paper is to furnish a group theory proof for the following two well known theorems: Each of the sums obtained by adding $p - 1$, $p - 2, \cdots, 2$ figurate numbers of orders $2, 3, \cdots, p - 1$ respectively is divisible by p whenever p is any odd prime number. Each of the coefficients except the first and the last in the expansion of $(a + b)^p$ is divisible by p. It is not claimed that the group theory proofs are easier than those usually given but they are based upon essentially different concepts and exhibit another element of contact between group theory and elementary number theory. The new proofs are based upon properties of the Sylow groups of order p^{p+1} contained in the symmetric group of degree p^2.

13. In this paper Professor Brand solves the systems composed of the differential equation of the elastic curve of a beam,

$$EIy'' = - M(x),$$

and the homogeneous boundary conditions $y(0) = 0, y'(0) = 0$ and $y(0) = 0$, $y(l) = 0$ by the use of Green's functions; he obtains two modified forms of the Fraenkel formula for flexural deflections which are particularly adapted to the study of statically indeterminate beams. After forming the solutions of the above equation with the non-homogeneous conditions $y(0) = \alpha$, $y'(0) = \alpha'$ and $y(0) = \alpha$, $y(l) = \beta$, the results are applied to determine the moments and reactions at the supports, and the deflections, of restrained and continuous beams under any system of loads. In particular, a very general form of Clapeyrou's theorem of three moments is obtained. The usual form of the Fraenkel formula, which has hitherto been deduced from the principles of mechanics, is shown to be a consequence of the above differential equation and its boundary conditions.

14. In this paper Professor Emch proves the theorem that every affinity in space, a collineation leaving the plane at infinity invariant, and defined in cartesian coordinates by

$$x' = a_0 + a_1x + a_2y + a_3z,$$
$$y' = b_0 + b_1x + b_2y + b_3z,$$
$$z' = c_0 + c_1x + c_2y + c_3z,$$

may be represented as the product of a rotation, a similitude, and a perspective affinity, or a homology with the center of homology at infinity. This is still true when the affinity is singular, so that all points (x, y, z) are transformed into points (x', y', z') of a certain plane. In this case the perspective affinity becomes a parallel projection. Based upon this theorem, it is not difficult to prove Pohlke's theorem and a number of related propositions. The demonstration of the theorem stated above, also for the case of a singular affinity, is necessary for a complete proof of Pohlke's theorem by this method.

The paper will appear in the *American Journal of Mathematics*.

15. Dr. Schweitzer constructs 10 classes of systems of postulates for a group, finite or infinite. The classes have the notations $\Sigma[\phi(x, y)]$, $\Sigma[f(x, y)]$, $\Sigma[\psi(x, y)]$, $\Sigma[\theta(x, y)]$, $\Sigma[\phi(x, y), f(x, y)]$, etc., where concretely $\phi(x, y) = xy$, $f(x, y) = xy^{-1}$, $\psi(x, y) = x^{-1}y$, $\theta(x, y) = x^{-1}y^{-1}$. To each of these undefined relations corresponds a fundamental iterative relation which may or may not be "self-conjugate"; in particular the relation for $\theta(x, y)$ is $\theta\{x, \theta(y, x)\} = \theta\{\theta(x, y), x\} = y$, which is self-conjugate. A commutator and transform satisfy certain quasi-transitive iterative functional equations. The preceding iterative relations are generalized. Iterative relations when the group is Abelian are also given. When the group is finite, among other sets of postulates two systems $\Sigma(xy^{-1})$, $\Sigma(x^{-1}y)$ are constructed analogous to Weber's system for a finite group and correspondingly two semi-groups "conjugate" to one another are found. A system $\Sigma[\theta(x, y)]$ contains the restriction that there exists uniquely an element x_0 such that $\theta(x_0, x_0) = x_0$. Under the class $\Sigma[\theta(x, y)]$ an analogue of the semi-group based on Weber's system is the "pseudo-group" with the properties (aside from unique closure) $\theta[\theta(x, y), x] = \theta[x, \theta(y, x)] = y$; $\theta(x', y) = \theta(x, y)$ implies $x' = x$; $\theta(x, y') = \theta(x, y)$ implies $y = y'$. Corresponding to the preceding systems of postulates for group, semi-group, pseudo-group, the author constructs postulational theories of fields. Illustrations from the theory of complex numbers are given.

16. The aim of Dr. Schweitzer's second paper is to describe in general logical terms the standard concepts of mathematical

independence. The author discriminates between an "invariant" and a "variable" type of independence of propositional functions $P_1, P_2, \cdots, P_n$ ($n = 2, 3, \cdots$) with variable components $R_1, R_2, \cdots, R_i$ ($i = 1, 2, \cdots$) of respective ranges $\rho_1, \rho_2, \cdots, \rho_i$. A second discrimination is that between "formal" and "material" independence. Subordinate to these discriminations is the characterization of the grade of independence by an index (n, k) $n \geqq k \geqq 0$ where k is the number of false propositions which arise when specific selections are suitably made from the variables $R_1, R_2, \cdots, R_i$. In this classification Professor E. H. Moore's "complete independence" appears as a variable independence of index (n, n) whereas the independence of Peano appears as an invariant independence of index $(n, 1)$.

In the second part of the paper the author calls attention to the use of "supernumerary" (non-independent) indefinables in primitive propositions as an instrument in the solution of problems of complete independence, i. e., if it is required to construct a completely independent set of postulates for a given logical domain the author would propose as a method for attaining this end a set of postulates involving supernumerary indefinables. Application to the author's postulates for an abstract group is made.

17. The paper of Professor Hildebrandt considers a number of generalizations of the theorem of Toeplitz which gives a necessary and sufficient condition under which if $\text{limit}_n\, x_n = x$ then

$$\text{limit}_n\, y_n = \text{limit}_n\, \Sigma_1^n a_{nm}\, x_m$$

is also x. These generalizations are a result of replacing the finite sum by an infinite sum, and considering under what conditions the sequence $\{y_n\}$ shall approach a limit, when the members of the sequence $\{x_n\}$ constitute a function belonging to various classes in the space of a denumerably infinite number of dimensions. The case in which x is a function of a continuous variable is also considered, the transformations from x to y being then on the basis of integration.

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