CREMONA'S WORKS.

Opere matematiche di Luigi Cremona. Pubblicate sotto gli auspici della R. Accademia dei Lincei. Volumes 1 and 2. Milano, Hoepli, 1914 and 1915. Quarto, viii + 492, 451 pp.

CREMONA'S career as geometer and teacher covered very nearly the second half of the nineteenth century. Born at Pavia in 1830, he was only eighteen when his ardent patriotism drew him into the war of independence. Returning after a year and a half, at the close of the war, he studied at the University of Pavia under Bordoni and Casorati and in 1853 took the laureate in civil engineering and architecture. After seven years of teaching in lower schools, he was called in 1860 to the University of Bologna as first professor of projective geometry and mechanics. After six years of intense activity there, he returned to Milan as a colleague of Brioschi at the Polytechnic and Normal School, training teachers in graphical statics for the technical institutes of the new Italy. From 1873 until the end of his life, 1903, he lectured in the University of Rome, on geometry, graphical statics, and "higher mathematics," meanwhile giving time and care without stint to the school of engineering, of which he was the founder and director. Usually also he gave courses in the normal department, to which he attached no less importance than to the more purely theoretical studies.

It may be doubted whether any great teacher has been actuated primarily by considerations of economic utility. While Cremona was intensely patriotic, it is evident from his writings that it was the innate love of his chosen science that moved him to teaching and to the preparation of the books through which his name is most widely known. In this collection the editors have not included his Elements of Projective Geometry; but we have his Introduction to a geometrical Theory of Plane Curves (1862), which was later translated (1865) into German by Curtze, and attained wide circulation and use; also his Fundamentals of a geometric Theory of Surfaces. This latter, combined with the memoir on Cubic Surfaces, was also most extensively known in Curtze's German version (1869). Here too are included, of course, the two epoch-making essays on transformations of plane curves (1863, 1865). It will be noticed that all these appeared during the Bologna period of six years. By way of comparison we note that Salmon's Higher Plane Curves first appeared in 1852, and that most of Steiner's work on algebraic curves was published between 1848 and 1854.

These two volumes* contain 78 titles, covering the author's publications down to 1868, from Pavia, Milan, and Bologna. Of these 78, 45 are dated from Bologna. Nine in the first, seven in the second volume, are answers to questions proposed in the *Nouvelles Annales*. Eight are book reviews, and neither dull nor devoid of significance; four are historical articles or addresses intended to stimulate geometrical study and research. One should read first of all the short peroration (volume 1, pages 252–3) of the inaugural address to his course on higher geometry at Bologna. Here occurs the often quoted climactic injunction: "Credite all' avvenire! questa è la religione del nostro secolo." But of more value to the young mathematician is the admonition: "Senza un' incrollabile costanza nella fatica non si giunge a possedere una scienza. Se questo nobile proposito è in voi, io vi dico che la scienza vi apparirà bella e ammiranda, e voi l'amerete così fortemente che allora in poi gli studi intensi vi riusciranno una dolce necessità della vita." There are not many such valuable sources of information and inspiration accessible to ambitious students.

Chasles was the type on whom at first Cremona modeled his own work. The geometrical purism of von Staudt was a later influence. Hence it is natural to find metric foundations for geometric definitions instead of, or interchangeably with, projective. The apparatus of algebraic geometry is built upon polars, and these upon distances. The geometric method is principally a use of terms or descriptive relations instead of equations. And the beginnings of enumerative geometry are here: we find questions of intersections or determining conditions for curves or surfaces treated by what has since become the principle of the Erhaltung der Anzahl, the axiom that the number persists even when the loci involved become specialized or degenerate. What Cremona is trying to attain and to diffuse is, not a critical knowledge of foundations, but an extensive knowledge of objects and theorems in projective geometry. This will doubtless be the

* The third and concluding volume has been published more recently (1917).

path followed by students for a long time to come; for it is easier to analyze and to discriminate than to unite discrete theories.

The Introduction contains an account quite complete, at its date, of plane cubic curves. It was to be desired, as Curtze said, that the theory of surfaces should cover those of the third order, and for this there was ample material in Cremona's Steiner prize essay. The depiction of a general cubic surface upon a plane, its rationality, was discovered independently by Cremona and Clebsch, and enabled the former to work out fully the geometry of algebraic curves upon the surface. This theme is more elaborately studied by R. Sturm in his well-known paper in the *Mathematische Annalen,* volume 21 (1883). A footnote there might suggest the notion that Cremona had committed a serious error in mentioning only one out of nine possible cases where the intersection of a cubic with a sextic surface breaks up into two twisted nonic curves. A reference to Cremona's own statement shows, however, that he was not attempting an exhaustive list of cases. This particular statement, like many in the same chapter, concludes with "etc., etc." That derogatory criticism was not in Sturm's intention is apparent also from the opening section of his "Nachruf" for his friend (*Archiv für Mathematik und Physik,* series 3, volume 8, 1904, pages 11–29).

In this Nachruf and in two others, Noether's in *Mathematische Annalen,* volume 59 (1904), pages 1–19, and Bertini's in *Proceedings of the London Mathematical Society,* series 2, volume 1 (1903–4), pages v–xviii, is found a clear and careful analysis of most of Cremona's important scientific papers. In particular, Sturm very neatly and fully analyzes the series of papers on twisted cubic curves, ten in number, which appeared from 1858 to 1864. The first two were devoted mainly to proving theorems, some twenty-five in number, which Chasles had published without proof in a note to his Aperçu historique, but both these and the later ones contain much that was original. The three that Chasles gives upon loci arising in the infinitesimal motion of a rigid body Cremona seems not to consider important enough to reproduce, as indeed they are but restatements of abstract theorems. The erroneous theorem of Chasles he does not criticize nor correct, namely that the locus of vertices of quadric cones passing through six arbitrarily fixed points in space is a twisted curve of the third order, whereas it is obviously two-dimensional and actually a Weddle's quartic surface.

One of these papers on twisted cubics had the experience, not unusual before the advent of the *Jahrbuch über die Fortschritte der Mathematik,* of being forgotten, and later duplicated. Böklen and F. Meyer in 1884 studied the question as new: When has a cubical parabola a directrix? If every osculating plane is perpendicular to two others that meet at a right angle, the locus of their common point is termed a directrix. When there is a directrix, what is its nature? The cubical parabola is a twisted cubic which osculates the plane at infinity, so that its other osculating planes trace out at infinity a conic. Orthogonality is conjugateness with respect to the absolute, an imaginary circle at infinity; so that the question concerns polar triangles of one conic circumscribed to another. These matters, and others related to them, had been thoroughly discussed in Cremona's final paper on twisted cubics in 1864, No. 50 of the present collection, particularly in § V. This fact probably escaped the notice even of Schroeter, whose writings were contemporaneous; for he does not mention it in his highly interesting paper: "Metrische Eigenschaften der cubischen Parabel," in *Mathematische Annalen,* volume 25 (1885). Such instances as this, of truths discovered yet insecurely fixed in the body of science, help one to realize the value of the Encyklopädie der mathematischen Wissenschaften.

Cremona himself was conscientious and indefatigable in searching out the work of his predecessors upon matters that he himself was investigating. Witness for example the voluminous list cited in his review of the translation of Amiot's geometry. This virtue is not too common, and we may be permitted to quote from Poncelet, who published with pardonable pride in volume 1 of his Traité (page 420) a letter from Dupin containing this tribute to that kind of honest dealing. "Je vous loue beaucoup d'avoir aussi rappelé honorablement les travaux de tous vos prédécesseurs; croyez-moi, cela n'ôte rien à votre mérite, et donne une haute idée de votre caractère. Vous prouvez par là que vous n'avez rien de commun avec cette école égoïste qui voudrait faire un monopole de la célébrité mathématique. Qu' ils maigrissent à leur gré de l'embonpoint d'autrui, et faites-les maigrir." Apropos, one reads with admiration the footnote (volume 2, page 18) wherein Cremona, after two pages of historical summary, excuses his previous ignorance of, and failure to cite, the works of Möbius and Seydewitz to which an essay of Schroeter had now directed his attention. He concludes with the phrase: "A présent je restitue unicuique suum." To be just is not easy, but he believed it to be a duty.

Of all these valuable works which constitute the visible monument of a valiant geometrician, those which will longest secure his fame are those numbered 40 and 62 in volume 2, upon geometric transformations of plane figures. Beyond the linear or projective transformations of the plane there were known the quadric inversions of Magnus, changing lines into conics through three fundamental points and those exceptional points into singular lines, to be discarded. Cremona described at once the highest generalization of these transformations, one-to-one for all points of the plane except a finite set of fundamental points. He found that it must be mediated by a net of rational curves; any two intersecting in one variable point, and in fixed points, ordinary or multiple, which are the fundamental points and which are themselves transformed into singular rational curves of the same orders as the indices of multiplicity of the points. When the fundamental points are enumerated by classes according to their several indices, the set of class numbers for the inverse transformation is found to be the same as for the direct, but usually related to different indices. Tables of such rational nets of low orders were made out by Cremona and Cayley, and a wide new vista seemed opening (such indeed it was and is) when simultaneously three investigators announced that the most general Cremona transformation is equivalent to a succession of quadric transformations of Magnus's type. This seemed a climax, and a set-back to certain expectations. But the fact remained, that elements do not constitute a theory, and that the generators of a group are no more important than its invariants. These invariants, for the most part, are yet to be determined, both synthetically and algebraically.

The expectation of geometrical inventors was turned next upon space of three dimensions; and there Cremona showed how a great variety of particular transformations can be constructed, but anything like a general theory is still in the future. When found, if within this century, such one-to-one transformations of three-dimensional space are certain to be hailed as Cremona transformations.

It is honorable alike to the Royal Academy of the Lincei and to the colleagues of Cremona and younger mathematicians, that they unite to preserve in worthy form the works of a justly celebrated scientist and leader. Eighteen already have shared the not inconsiderable labor of thorough editing, and their corrections and explanatory notes, appended to each volume, form a valuable aid to the reader. The highest tribute that can be paid to the memory of a scientist is the labor that makes his work more useful to the next generation.

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BLICHFELDT'S COLLINEATION GROUPS.

Finite Collineation Groups. By H. F. BLICHFELDT. Chicago, University of Chicago Press, 1917. 12mo. 12 + 194 pp.

THIS little volume forms a notable contribution to the series of mathematical texts by American authors that have appeared in recent years. Coming from the pen of an author who has an unusual mastery of his subject, it is moreover almost unique in its field, the promised text by Wiman for the Teubner series (as far as the reviewer is aware) not having appeared. Certain parts of the subject, particularly the theorems depending on the invariance of a Hermitian form and the theory of group characteristics, may be found in the second edition of Burnside's Theory of Groups, which appeared in 1911. A considerable part also of the material in the present treatise may be found in Part II of Finite Groups, by Miller, Blichfeldt, and Dickson, which was written by the same author.

On the other hand there is much in the present volume that cannot be found elsewhere except in scattered journal articles, and some of the results at the close of Chapter IV seem to be entirely new. The author's own share in the development of the subject is a very notable one, the theorems in Chapter IV concerning the linear groups in n variables being almost entirely his own. In addition the complete determination of the groups in three and four variables was first made by him, the earlier work along this line being reproduced in Chapters V and VII in a somewhat revised form. There are