CELEBRATED PROBLEMS OF GEOMETRY.

Sur les Problèmes célèbres de la Géométrie élémentaire non résolubles avec la Règle et le Compas. Par F. GOMES TEIXEIRA. Coïmbre, Imprimerie de l'Université, 1915.* 4to. 132 pp.

THERE have been many historical surveys of the three famous problems of the ancients. One such was Montucla's anonymous work of 1754 on the history of the problem of the squaring of the circle with a supplement concerning the problems of the duplication of the cube and the trisection of an angle.† But a more adequate history of the problem of the duplication of the cube was published by Reimer in 1798.‡ An accurate and still more elaborate presentation which took due account of later research was published about a century later by Ambros Sturm.§ A. Conti's account of the problems of duplication of the cube and trisection of an angle occupied about 70 pages of the second part of Enriques's Fragen der Elementargeometrie, which appeared in 1907. This same work contained B. Calò's chapter (60 pages) on transcendental problems, especially that of squaring the circle. These chapters underwent some revisions in the new Italian edition: Questioni riguardanti le Matematiche elementari (1914).‖ Prior to Calò's article, one of the best sketches of the history of the problem of squaring the circle was by Rudio, 1892;¶ Vahlen's discussion** (1911) is also valuable; Hobson's most readable history appeared in 1913.

Such are some of the chief historical surveys. The most

* Also published as an Appendix to Gomes Teixeira's Obras sobre Mathematica, vol. 7, 1915, pages 285–412.

† Histoire des recherches sur la quadrature du cercle. Paris, MDCCLIV. 47 + 304 pp. + 8 plates. Nouvelle édition revue et corrigée (par S. F. Lacroix). Paris, 1831. 16 + 300 pp. + 4 plates.

‡ N. T. Reimer, Historia problematis de cubi duplicatione. Gottingæ, MDCCXCVIII. 16 + 222 pp. + 2 plates.

§ A. Sturm, Das Delische Problem, Linz, 1895–1897. 140 pp.

‖ Vol. 2, Bologna, 1914. Pp. 185–335.

¶ F. Rudio, Archimedes, Huygens, Lambert, Legendre. Vier Abhandlungen über die Kreismessung. Deutsch herausgegeben und mit einer Uebersicht über die Geschichte des Problemes von der Quadratur des Zirkels, von den ältesten Zeiten bis auf unsere Tage. Leipzig, 1892. History, pp. 1–69.

** T. Vahlen, Konstruktionen und Approximationen. Leipzig, 1911. Pp. 175 ff. and 306 ff.

recent bibliography of the problems is by Professor Guimaraes.* In the writer's opinion the most elementary presentation of the proofs of the impossibility of their solution with ruler and compasses is due to Klein.†

Since Montucla's work is very scarce, those unacquainted with German or Italian who wished to learn the main facts in such surveys as the ones to which I have referred, have had, till recently, considerable difficulty in satisfying their desires. We have now, however, the very interesting and excellent volume under review, of Professor Gomes Teixeira, Rector of the University of Porto. His power of lucid exposition and his scholarly style are probably familiar to many Americans through the two-volume Traité des Courbes spéciales remarquables planes et gauches of 1908–09.‡

Nearly the whole of the volume on "problèmes célèbres" is given over to a consideration of the three famous problems of the ancients. Chapter I (pages 5–46) is entitled: "Sur le problème des moyennes proportionnelles. Duplication du cube;" Chapter II (pages 47–82): "Sur la division de l'angle;" Chapter III (pages 83–104): "Sur la quadrature du cercle;" and the last chapter: "Sur l'impossibilité de la résolution par la règle et le compas des problèmes considérés précédemment." There are many references to the author's treatise on curves and it is especially in this connection that new features are introduced.

For example, in the first chapter we have: the curve of Archytas—a skew curve; the kampyle of Eudoxus, the simple folium in the method of Villapandus, and the conchoid of Nicomedes—quartics; the hyperbola mesolabica of Viviani, the circular unicursals in the solutions of Plato and Diocles, and the right strophoid in connection with Huygens's solution—cubics; the method of Menæchmus—by conics; and so on. The chapter contains also solutions by Hero of Alexandria, Philo of Byzantium, Apollonius, Eratosthenes, Viète, Descartes, Fermat, Newton, Clairaut, and Montucci.

The second chapter sets forth the methods of Hippias, Archimedes, Nicomedes, Pappus, Etienne Pascal (with his

* R. Guimaraes, "Algunas consideraciones sobre tres problemas célebre de geometría elemental," *Revista de la Sociedad matemática Española*, año 6, Enero-Abril, 1917, pp. 18–27, 74–94.

† F. Klein, Vorträge über ausgewählte Fragen der Elementar-Geometrie. Leipzig, 1895. English translation by Beman and Smith, Boston, 1897.

‡ Pages 1–284 of Gomes Teixeira's Obras sobre Mathematica, vol. 7 (1915), contain five chapters supplementary to this work.

limaçon), Descartes and Fermat, Kinner, Ceva, Maclaurin, Delanges, Chasles, Lucas, Catalan, Longchamps, and Kempe, with many interesting connections and generalizations. It is shown that the solution of the following problem of Archimedes reduces to that of the trisection of an angle: "To cut a sphere by a plane so that the volumes of the segments are to one another in a given ratio." At this point it would have been interesting to have added a reference to Brocard's pamphlet, Mémoire sur divers problèmes de géométrie dont la solution dépend de la trisection de l'angle (Algiers, 1912).

Viète and Descartes stated that the solution of any problem depending on an equation of the third degree could be reduced to the solution of a problem of finding two mean proportionals, or to that of the trisection of an angle. With a proof of this theorem, and some general remarks, the chapter concludes.

"Given a fixed conic (except circle and line pair) in the plane of construction every problem of the third order can be carried through with ruler and compasses." Professor Gomes Teixeira has apparently followed Vahlen in crediting this theorem to "S. Smith" (1868). To the Englishman, "H. J. S. Smith" seems more natural.

"All problems of the third order can be carried through with ruler alone if a complete fixed curve of the third order is given; for metrical problems a square (or rectangle) must also be given." This theorem was shown by London in 1896. Its statement on page 82 needs to be revised. Compare Conti's article. In his Arithmetica Universalis Newton solved cubic and biquadratic equations by means of the conchoid and ruler and compasses.

In the third chapter there are a number of unusual expressions for π, and its powers, taken from the writings of Wallis, Euler, and Cauchy. Due credit is given to Chinese discoveries in accordance with Mikami's History. On page 88, line 14, for Chang Hing read Chang Hêng.*

The derivation of the results leading to the fundamental theorem by means of which it is shown that the problem of the duplication of the cube is impossible is based mainly on the discussion in Petersen's Theory of Equations. The work concludes with Klein's proof of the impossibility of the

* Some fairly obvious misprints occur at the following places: page 25, line 4 from bottom; page 26, line 17; page 56, line 12; page 68, line 6 from bottom; page 82, line 6; page 103, last line; page 105, line 6 from bottom; page 122, line 25.

problem of squaring the circle with ruler and compasses.

On page 122 occurs the sentence: "La divisionde la circonference en 3 et 5 parties égales a été considérée dans les Elements d'Euclide." Quite true; but why not have written "3, 4, 5, 6 et 15 parties égales"? The statement that the first geometric construction of the regular polygon of 17 sides was found by Erchinger needs revision. Gauss reported Erchinger's paper in 1825 and pointed out that its merit was not so much in the construction as in the synthetic proof of its correctness. Indeed Gauss himself refers to two earlier constructions by Paucker.* At least two more were published before 1825; one by John Lowry in 1819† and the other by Samuel Jones in 1820.‡

We heartily recommend Professor Gomes Teixeira's book for every mathematical library, as no other publication of the kind can take its place. The little book is characterized by marked individuality. When a new edition is called for we hope that the author may be moved to add another chapter on still more of the many famous problems of the category he has been considering. For example, an adequate history of the following century-old problem has not yet been published: "Given the length of the bisectors of the angles of a triangle between the vertices and the opposite sides to construct the triangle." In 1911, Professor R. P. Baker published a hundred-page doctor's dissertation on this problem.

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* (*a*) "Geometrische Verzeichnung des regelmässigen 17-Ecks und 257-Ecks in d. Kreis," *Jahresverhandl. d. kurländische Gesellschaft für Literatur und Kunst*, Mitau, Band 2, 1822. (*b*) Die ebene Geometrie der geraden Linie und des Kreises, Königsberg, 1823, p. 187.

† *The Mathematical Repository*, new series, vol. 4 (1819), p. 160.

‡ The paper dated "Dublin, 17th October, 1819" and read January 24, 1820, was published in *Transactions of the Irish Academy*, vol. 13 (1818), pp. 175–187.