stituent of degree m must have in this maximal subgroup at least one transitive constituent the degree of which is a divisor ($> m$) of $m(m - 1)$. This paper has been offered to the *Transactions* for publication.

5. In this paper Professor Winger shows how the classical properties of the rational cubic, R^3, can be derived quite simply from the theory of involution. The method is then employed in the discovery of new theorems. In particular the contact conics, including the perspective conics, are discussed. The paper closes with some theorems on the hyperosculating curves, i. e., curves whose complete intersections with R^3 fall at a point.

W. A. MANNING,
Secretary of the Section.

ON INTEGRALS RELATED TO AND EXTENSIONS OF THE LEBESGUE INTEGRALS.

BY PROFESSOR T. H. HILDEBRANDT.

(*Continued from page 144.*)

III. STIELTJES INTEGRALS AND THEIR GENERALIZATIONS.

While the Lebesgue integral received almost immediate attention and recognition and found its way rapidly into mathematical literature and thought, it is only recently that the definition of Stieltjes seems to have received the consideration to which it is entitled by virtue of its range of applicability and usefulness. As a matter of fact, in the opinion of the writer, it seems to be destined to play the central rôle in integrational and summational processes in the future.

1. *Definition of the Stieltjes Integral.*—(Cf. Stieltjes (23), pages 71 ff.; Perron (17), page 362; Fréchet (5), pages 45–54; Young (29), pages 131, 137.) A definition for this integral was given first by Stieltjes in his memoir on continued fractions. The integral depends for its value upon two functions $f(x)$ and $v(x)$ defined on an interval (a, b). We suppose that they are both bounded. Then the definition is as follows:

DEFINITION. *Divide (a, b) into a finite number of intervals by the points $a = x_0, x_1, \cdots, x_n = b$. Let ξ_i be any point interior to (x_{i-1}, x_i) and form the sum*

$$s = \sum_{i=1}^{n} f(\xi_i)(v(x_i) - v(x_{i-1})).$$

If this sum has a limit as the number of divisions is increased and their maximum length diminished, this limit is the Stieltjes integral

$$\int_a^b f(x)dv(x).$$

Stieltjes stated his definition for $f(x)$ any continuous function and $v(x)$ a monotonic non-decreasing function and showed that in this case the integral exists. If we desire this integral to exist for every function continuous on (a, b) it is necessary and sufficient that $v(x)$ be of bounded variation. However for the existence of an $\int fdv$ it is not necessary either that $f(x)$ be continuous nor that $v(x)$ be of bounded variation. For instance, we have the proposition

(1) If $f(x)$ is continuous and $v(x)$ of bounded variation then the Stieltjes integral

$$\int_a^b vdf$$

exists also and we have

$$\int_a^b vdf = \int_a^b fdv + f(b)v(b) - f(a)v(a).$$

The proof depends upon the rearrangement of the terms in the sum s.*

In the sequel we shall confine ourselves mainly to the case in which v is a function of bounded variation, or more particularly monotonic non-decreasing, from which the former case can be deduced on account of the fact that every function of bounded variation can be expressed as the difference of two monotonic non-decreasing functions.

As in the case of a Riemann integral we have the following theorem relative to the existence of a Stieltjes integral, v being a function of bounded variation:

* Cf., for instance, Bliss (1), p. 29.

(2) *A necessary and sufficient condition for the existence of the Stieltjes integral* $\int f dv$ *is that the total variation* of the function* $v(x)$ *over the points at which* $f(x)$ *is discontinuous shall be zero.*

Among other things, this theorem would require that the function v be continuous at points of discontinuity of f.

The following simple instances of Stieltjes integrals may be of interest:

(*a*) If there exists a function $w(x)$ such that

$$v(x) = \int_a^x w(x)dx,$$

then

$$\int_a^b f(x)dv(x) = \int_a^b f(x)w(x)dx,$$

the integral on the right-hand side being taken in the same sense as that of $w(x)$.

(*b*) Suppose $v(x)$ monotonic non-decreasing, and discontinuous at the points $x_1 \leqq x_2 \leqq x_3 \cdots$, which approach b as a limit; let the measure of the discontinuities of $v(x)$ at these points be the positive numbers $v_1, v_2, \cdots$, respectively. Of necessity the series $\Sigma_n v_n$ is convergent. In the interior of the interval $(x_{n-1} \leqq x < x_n)$ let $v(x)$ be constant and equal to $\sum_{i=1}^{n-1} v_i$, and $v(b) = \Sigma_n v_n$. Then evidently, if $f(x)$ is continuous in (a, b),

$$\int_a^b f(x)dv(x) = \Sigma_n v_n f(x_n).$$

(*c*) If in example (*b*) we assume that in the interval $(x_{n-1} \leqq x < x_n)$, we have

$$v(x) = x + \sum_{i=1}^{n-1} v_i,$$

with $v(b) = b + \Sigma_n v_n$, then

$$\int_a^b f(x)dv(x) = \int_a^b f(x)dx + \Sigma_n v_n f(x_n).$$

These examples illustrate the fact that the Stieltjes integral

* For the definition of the total variation over a set of points cf. below, § 5. This theorem has been communicated to me by Bliss. Cf. also Young (29), pp. 132, 133, for the case where $v(x)$ is monotonic.

in addition to giving an ordinary definite integral introduces also the values of the function integrated at special points. In this direction Fréchet has shown

(3) The Stieltjes integral $\int_a^b f(x)dv(x)$ can be broken up into the sum of three terms

$$\int_a^b f(x)dv(x) = \int_a^b f(x)a(x)dx + \Sigma_n v_n f(x_n) + \int_a^b f(x)du(x),$$

in which $a(x)$ is a summable function and hence the first term a Lebesgue integral; the x_n are the points of discontinuity of $v(x)$ and $v_n = v(x_n + 0) - v(x_n - 0)$; and u is a continuous function of bounded variation which has a derivative zero excepting at a set of measure zero.

For if we set

$$\varphi(x) = \sum_{a < x_n \leqq x} (v(x_n) - v(x_n - 0)) + \sum_{a \leqq x_n < x} (v(x_n + 0) - v(x_n)),$$

then the function $v(x) - \varphi(x)$ is continuous and of bounded variation. Hence, if $a(x)$ is equal to one of its derived functions wherever this is finite, and zero everywhere else, then

$$v(x) - \varphi(x) = \int_a^x a(x)dx + u(x),$$

where $u(x)$ is the variation of $v(x) - \varphi(x)$ over the set of points of measure zero at which the derivative $a(x)$ would be infinite.* The form of the theorem is then an immediate consequence of this decomposition of the function $v(x)$.

2. *Some Properties of the Stieltjes Integral.*—(Cf. Perron (17), pages 366 ff.; Riesz (21), pages 38, 39.) We note the following properties of the Stieltjes integral, v being a function of bounded variation and f being supposed to be integrable with respect to v in the interval (a, b):

(1) $\int_a^c fdv + \int_c^b fdv = \int_a^b fdv.$

(2) $\int_a^b (f + g)dv = \int_a^b fdv + \int_a^b gdv.$

* Cf. Bliss (1), p. 40; de la Vallée Poussin (24), p. 93.

(3) $\int_a^b cfdv = c\int_a^b fdv.$

(4) If $f \geqq 0$ and $v(x)$ is monotonic non-decreasing then

$$\int_a^b fdv \geqq 0.$$

(5) $\left|\int_a^b fdv\right| \leqq \int_a^b |f|\,|dv| = \int_a^b |f|du,$ where u is the total variation of $v(x)$ from a to x.

(6) $\int_a^b dv = v(b) - v(a).$

(7) If $\lim_n f_n(x) = f(x)$ uniformly on (a, b), then

$$\lim_n \int_a^b f_n(x)dv = \int_a^b f(x)dv.$$

(8) *If* $\int_a^b fdv = 0$ *for all continuous functions and* v *is a function of bounded variation, then* $v(x)$ *is constant except at a denumerable set of points between* a *and* b, *and conversely.*

3. *Comparison of the Stieltjes Integral and the Lebesgue Integral.*—(Cf. Lebesgue (14); Van Vleck (25).) Lebesgue (14) has shown that every Stieltjes integral is expressible as a Lebesgue integral. He gives two modes of procedure. In the first let $w(x)$ be the total variation of $v(x)$ in the interval (a, x). Let $x(w)$ be the inverse function of w, with the convention that in case $w(x)$ is constant in the interval (l, m), we take for $x(w)$ any of the values in this interval, for instance l. Substitute this function in the function $v(x)$, and assume that if x_0 is a point of discontinuity of $v(x)$ then we make $v(x(w))$ linear in the intervals between $w(x_0 - 0)$ and $w(x_0)$, and $w(x_0)$ and $w(x_0 + 0)$. Then $v(x(w)) = u(w)$ will be a function of bounded variation having total variation equal to w at any point, and so, if u' is any of the derivatives of u with respect to w, it will take only the values $+ 1$ and $- 1$, and we shall have

$$u(w) = \int_0^w u'(w)dw.$$

Hence

$$\int_a^b f(x)dv(x) = \int_0^{w(b)} f(w)u'(w)dw,$$

where now $f(w)$ is no longer continuous but has a denumerable set of discontinuities of the first kind, i. e., $f(w_0 - 0)$ and $f(w_0 + 0)$ exist at every point; and the integral on the right is a Lebesgue integral.

The other transformation which Lebesgue suggests rests upon the fact that a function of bounded variation can be expressed as the difference of two monotonic non-decreasing functions. If $v(x) = g(x) - h(x)$ and we define $x(g)$ and $x(h)$ as above, and then replace the variables g and h by

$$g - g(a) = (g(b) - g(a))t = K_1 t,$$
$$h - h(a) = (h(b) - h(a))t = K_2 t$$

and set $x(g) = x_1(t)$ and $x(h) = x_2(t)$, then we can express

$$\int_a^b f(x)dv(x) = \int_0^1 [K_1 f(x_1(t)) - K_2 f(x_2(t))]dt.$$

We note that in the Fréchet expression of § 1, the interval of variation of f is still (a, b); in both of these last expressions we have a new variable of integration and a new interval of integration.

On the other hand Van Vleck (25) has given a very simple transformation of a Lebesgue integral into a Stieltjes integral. As a matter of fact we might point out that the *Lebesgue integral is by its very manner of definition a Stieltjes integral.*

He shows that if $l < f(x) < L$ and $\mu(y)$ is defined to be the measure of the set E for which $l \leqq f < y$, then

$$(L)\int_a^b f(x)dx = (S)\int_l^L y d\mu(y).$$

By using the integration by parts formula (1), Bliss ((1), page 28) shows that a Lebesgue integral is reducible to a Riemann integral,* viz.,

$$(L)\int_a^b f(x)dx = (S)\int_l^L y d\mu(y) = L(b - a) - (R)\int_l^L \mu(y)dy.$$

* Cf. also Young (26), pp. 245ff.; Denjoy (4(*b*)), pp. 191–203; Lamond, *American Journal of Mathematics*, vol. 36 (1914), pp. 387, 388, in which the final formulas are incorrect.

If $f(x)$ is not bounded on (a, b), we have

$$(L) \int_a^b f(x)dx = \int_{-\infty}^{\infty} y d\mu(y),$$

where

$$\int_{-\infty}^{\infty} y d\mu(y) = \lim_{\alpha,\, \beta \doteq \infty} \int_{-\alpha}^{\beta} y d\mu(y).$$

A similar reduction of the Pierpont integral to the Stieltjes, and thence to a Riemann integral can be made, if we express the measure function $\mu(y)$ in terms of upper measure instead of measure.

4. *Applications.*—(Cf. Riesz (21), pages 40–43, 51–54; (22).) We would seem to have shown in the last section that the Stieltjes and Lebesgue integrals are equivalent. This is true in the sense that each of them can by a transformation be evaluated in terms of the other, but in the transformation we change both the function and the interval of integration. It does not however follow from this that wherever we use a Stieltjes integral a Lebesgue integral will serve just as well. Perhaps the best illustration of this fact are the two applications of Stieltjes integrals which we shall briefly consider, the first of which has probably contributed more than any other to direct attention to the Stieltjes integral.

The first application is in the theory of linear functional operations. Let $\mathfrak{F}$ be the class of all continuous functions f on an interval (a, b). Then U is said to be a functional operation on $\mathfrak{F}$, if for every function f of $\mathfrak{F}$ there is a corresponding real number $U(f)$.

A functional operation which is *linear* is generally defined to have the following two properties:

(*a*) *distributivity*: $U(f_1 + f_2) = U(f_1) + U(f_2)$;

(*b*) *continuity*: if $\lim_n f_n = f$ uniformly on (a, b), then $\lim_n U(f_n) = U(f)$.

It can then be shown that every linear functional operation U possesses also the properties

(*c*) $U(cf) = cU(f)$, c being any constant;

(*d*) there exists a quantity M such that, for every function f of the class $\mathfrak{F}$, $|U(f)| \leqq M \times$ maximum of $|f|$.

If the functional operation U has the property (*a*), then the property (*d*) is equivalent to the continuity property (*b*).

As illustrations of linear functional operations we cite

$$U(f) = \int_a^b f(x)u(x)dx,$$

$$U(f) = \Sigma_n u_n f(x_n),$$

where $u(x)$ is any summable function on (a, b), and where the u_n constitute an absolutely convergent series and the x_n are any points of the interval (a, b).

Hadamard* has shown that for every linear functional operation $U(f)$ on the class of all continuous functions there exists a sequence of integrable functions $u_n(x)$ such that

$$U(f) = \lim_n \int_a^b f(x)u_n(x)dx,$$

but the functions $u_n(x)$ do not necessarily have a summable function as a limit. On the other hand, Helly† has shown that there exist a set of constants $u_m^{(n)}$, a set of points $x_m^{(n)}$ on the interval (a, b), and a set of integers r_m, such that

$$U(f) = \lim_{m,\, r \doteq \infty} \sum_{n=1}^{r_m} u_m^{(n)} f(x_m^{(n)}).$$

Each of these expressions for a linear $U(f)$ is rather complicated. Perhaps the most elegant expression for a linear operation has been given by Riesz, who shows that there exists a function of bounded variation $u(x)$ such that

$$U(f) = \int_a^b f(x)du(x).$$

Riesz's second proof of this fact (cf. (22)) is really quite simple and elegant. It is made to depend upon the extension of the application of $U(f)$ to functions which are constant on intervals, and this leads directly to a definition of the function $u(x)$ and to the expression of $U(f)$ for any continuous function $f(x)$ as a Stieltjes integral.

A second application is allied to the Weierstrass theorem that every continuous function is expressible as the limit of a uniformly convergent sequence of polynomials, i. e., linear combinations of the functions $1, x, x^2, \cdots, x^n, \cdots$. This

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* Cf. Leçons sur le Calcul des Variations, I, pp. 289–302.

† Cf. *Wiener Berichte*, vol. 121_{2a} (1912), pp. 277–8.

theorem has suggested the question: under what conditions, necessary and sufficient, can we say of a given set of functions $\varphi_1(x), \cdots, \varphi_n(x), \cdots$, continuous on (a, b), that any continuous function can be expressed as the limit of a uniformly convergent sequence of functions which are linear combinations of the $\varphi_n(x)$.

Various necessary or sufficient conditions in terms of Riemann integrability and Lebesgue summability were given, involving the derivatives of the functions φ_n, but it remained for the Stieltjes integral to provide a condition both necessary and sufficient, which is as follows (cf. Riesz (21), pages 51–54):

A necessary and sufficient condition that every continuous function may be uniformly approximated by linear combinations of a set of functions $[\varphi_1(x), \cdots, \varphi_n(x), \cdots]$ is that the only solution of the equations

$$\int_a^b \varphi_n(x)du(x) = 0 \qquad (n = 1, 2, \cdots),$$

for a $u(x)$ which for every x_0 satisfies the condition

$$u(x_0) = \tfrac{1}{2}(u(x_0 + 0) + u(x_0 - 0)),$$

is $u(x) = \text{constant}$.

5. *Extension of the Stieltjes Integral.*—In constructing a descriptive definition of integration Lebesgue* sets down the following six properties sufficient to characterize his definition of integration:

$L(1)$ $\displaystyle\int_a^b f(x)dx = \int_{a+h}^{b+h} f(x - h)dx.$

$L(2)$ $\displaystyle\int_a^b f(x)dx + \int_b^c f(x)dx + \int_c^a f(x)dx = 0.$

$L(3)$ $\displaystyle\int_a^b f(x)dx + \int_a^b \varphi(x)dx = \int_a^b (f(x) + \varphi(x))dx.$

$L(4)$ If $f \geqq 0$ and $b > a$, then $\displaystyle\int_a^b f(x)dx \geqq 0.$

$L(5)$ $\displaystyle\int_0^1 1 \times dx = 1$ or $\displaystyle\int_a^b 1 \times dx = b - a.$

* Cf. (13), pp. 98, 99. Cf. also *Annales de l'Ecole Normale Supérieure*, ser. 3, vol. 27 (1910), pp. 368, 369 and 374 ff.

$L(6)$ If $f_n(x)$ is a monotonic non-decreasing sequence of functions having $f(x)$ as a limit, then

$$\lim_n \int_a^b f_n(x)dx = \int_a^b f(x)dx.$$

If we turn back to § 2, we observe that there are properties of the Stieltjes integral which are very similar to these; in particular we find a marked similarity between (1), (2), (4) and (6) of § 2 and $L(2)$, $L(3)$, $L(4)$, and $L(5)$ above. We observe that the main dissimilarity between the properties are in the absence of an analogue of $L(1)$ and the fact that in the analogue (7) of $L(6)$ the convergence to the limiting function $f(x)$ is uniform instead of monotonic. In discussing this last difference first, we observe that it is really *the property* $L(6)$ *which characterizes the Lebesgue integral as distinct from the Riemann,* i. e., in effect that the continuity of the Riemann integral operation is one based on uniform convergence of the functions, and that of the Lebesgue one is based on monotonic sequences. It is this distinction which we find emphasized in the later treatments of Young (cf. (27) and (28)). It seems that an obvious conclusion is that, if we start from the following properties modelled after those of Lebesgue, we might expect to arrive at a Lebesgue generalization of the Stieltjes integral:

$S(1)$ $v(x)$ is a monotonic non-decreasing function of x.

$S(2)$ $\int_a^b f(x)dv(x) + \int_b^c f(x)dv(x) + \int_c^a f(x)dv(x) = 0.$

$S(3)$ $\int_a^b (f(x) + \varphi(x))dv(x) = \int_a^b f(x)dv(x) + \int_a^b \varphi(x)dv(x).$

$S(4)$ If $f \geqq 0$ and $b > a$, then $\int_a^b f(x)dv(x) \geqq 0.$

$S(5)$ $\int_a^b dv(x) = v(b) - v(a).$

$S(6)$ If f_n is a monotonic non-decreasing sequence of functions such that $\lim_n f_n = f$, then

$$\lim_n \int_a^b f_n(x)dv(x) = \int_a^b f(x)dv(x).$$

In the above properties $S(1)$ is not an analogue of $L(1)$, which would have the form

$$\int_a^b f(x)dv(x) = \int_{a+h}^{b+h} f(x-h)dv(x).$$

If we use this property in connection with $S(5)$ we get

$$v(b') - v(a') = v(b'') - v(a'')$$

for every pair of subintervals of (a, b) for which

$$b' - a' = b'' - a''.$$

In as much as v is a continuous function except at a denumerable infinity of points, this would require that the function $v(x)$ be continuous throughout the interval, and from this in turn we would conclude that $v(x)$ is linear and of the form $cx + d$. If we impose on this the requirement $L(5)$ instead of $S(5)$ we get $v(x) = x + c$. In other words, it seems that the property $L(1)$ plays a principal rôle in characterizing the Lebesgue and Riemann integrals as distinct from the Stieltjes.

By following through a method of reasoning similar to that of Lebesgue ((13), pages 98–102), but on the basis of the properties $S(1)$–$S(6)$, we arrive at a result similar to the one which he obtains, viz.,

In order to find $\int_a^b fdv$, it is sufficient to know how to find $\int_a^b \psi dv$, where ψ is a function which takes only the values zero and unity.

Lebesgue determines his function ψ to be the measure of the set of points E for which $\psi = 1$. Radon ((19), pages 1305 ff.) has suggested a method of procedure to be applied to any monotonic non-decreasing function which has the continuity property $v(x - 0) = v(x)$ at every point of the interval. This requirement is a result of the fact that he considers as the basis of his operations half open intervals $(a \leqq x < b)$. We modify this method slightly, using open intervals instead of half open ones.

We proceed to define a function $v(E)$ on a set of points E corresponding to the measure function, as follows: If $E = (a' < x < b')$ then $v(E) = v(b' - 0) - v(a' + 0)$. If a and b are the extremities of the fundamental interval, then we take for $E = (a \leqq x \leqq b)$, $v(E) = v(b) - v(a)$. In order to avoid circumlocution, we might think of the interval (a, b) as extended by ϵ on each end, and have $v(a - \epsilon) = v(a)$ and $v(b + \epsilon) = v(b)$, and think of the interval (a, b) as enclosed in an open interval extending beyond a and b. Complementary sets will be taken with respect to the *closed* interval (a, b).

For any set E, we define then a quantity $\bar{v}(E)$ as follows: Enclose the points of E in a finite or denumerable infinitude of open intervals α_n; then

$$\bar{v}(E) = \overline{B}(\Sigma_n v(\alpha_n)),$$

i. e., $\bar{v}(E)$ is the least upper bound of the sums $\Sigma_n v(\alpha_n)$ for all possible enclosures of the set E in open intervals. Let CE be the complementary set to E relative to the closed interval (a, b). Then we define

$$\underline{v}(E) = v(a, b) - \bar{v}(CE).*$$

When $\bar{v}(E) = \underline{v}(E)$, we say that E is *measurable relative to* $v(x)$, and define $v(E)$ to be the common value. The totality $\mathfrak{E}$ of sets measurable relative to $v(x)$ form a class of sets which have the following properties:†

(1) If E_1 and E_2 belong to the class $\mathfrak{E}$, then $E_1 + E_2$ and E_1E_2 also belong to $\mathfrak{E}$.

(2) If $E_1, \cdots, E_n, \cdots$ belong to $\mathfrak{E}$ and are mutually distinct, then $\Sigma_n E_n$ also belongs to $\mathfrak{E}$.

(3) The Borel measurable sets belong to $\mathfrak{E}$.

Further $v(E)$ has the properties

(1) If E_1 and E_2 belong to $\mathfrak{E}$, then

$$v(E_1 + E_2) + v(E_1E_2) = v(E_1) + v(E_2).$$

(2) If $E_1, \cdots, E_n, \cdots$ belong to $\mathfrak{E}$ and are mutually distinct, then

$$v(\Sigma_n E_n) = \Sigma_n v(E_n),$$

i. e., v is what Radon ((19), page 1299) calls an *absolutely additive function.*

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* Note that the values of $\bar{v}(E)$ and $\underline{v}(E)$ are independent of the values of $v(x)$ at its points of discontinuity. We might then have assumed with Radon $v(x - 0) = v(x)$ at every point. In particular we have

$$v(E \equiv a' \leqq x \leqq b') = v(b' + 0) - v(a' - 0), \text{ not } v(b') - v(a').$$

† For the proof cf. Radon (19), pp. 1305 ff. Bliss (1), pp. 12–17, has given a careful analysis of the case in which $v(x)$ is a continuous monotonic non-decreasing function.

(3) For every E of $\mathfrak{E}$ and for every $\epsilon > 0$, there exists a closed set E', contained in E and $\mathfrak{E}$, such that $v(E) - v(E') < \epsilon$.

In case $v(x)$ is of bounded variation instead of monotonic non-decreasing, it can be expressed as the difference of two monotonic non-decreasing functions $p(x)$ and $n(x)$. We can then find the class $\mathfrak{E}_1$ of all sets measurable relative to $p(x)$ and $\mathfrak{E}_2$ of all sets measurable relative to $n(x)$. We say that the class $\mathfrak{E}$ of all sets measurable relative to v is the greatest common subclass of the classes $\mathfrak{E}_1$ and $\mathfrak{E}_2$, and set

$$v(E) = p(E) - n(E).$$

The *total variation of v over the set E* is defined to be

$$t(E) = p(E) + n(E).$$

6. *The Fréchet Generalized Integral.*—(Cf. Fréchet (6).) Instead of proceeding to the definitions of integration of the function f with respect to the $v(E)$ defined in the preceding section, on an interval (a, b) or a set E, we prefer to discuss the definition of integration suggested by Fréchet, which includes the Lebesgue, Young, Pierpont, and Stieltjes integrals as special cases by properly assigning the function v.

Suppose a set $\mathfrak{P}$ of general elements p. Suppose further a class $\mathfrak{E}$ of subsets E of elements of p, which has the following properties:

(1) If E_1 and E_2 belong to $\mathfrak{E}$, then $E_1 + E_2$ and E_1E_2 also belong to $\mathfrak{E}$.

(2) If $E_1, \cdots, E_n, \cdots$ are sets belonging to $\mathfrak{E}$, which are without common elements, then $\Sigma_n E_n$ also belongs to $\mathfrak{E}$.

On the class $\mathfrak{E}$, we suppose that there is defined an *absolutely additive function* $v(E)$ which has the following property: If $E_1, \cdots, E_n, \cdots$ are sets belonging to $\mathfrak{E}$ without common elements, then

$$v(\Sigma_n E_n) = \Sigma_n v(E_n).$$

Then it can be shown that there exist two functions $v_1(E)$ and $v_2(E)$, satisfying the same condition as v and in addition the condition $v_1(E) \geqq 0$ and $v_2(E) \geqq 0$ for every E, and such that

$$v(E) = v_1(E) - v_2(E).$$

We shall assume in the sequel that in addition to being absolutely additive, the function $v(E)$ is also *monotonic*, i. e.,

$$v(E) \geqq 0 \text{ for every } E.$$

The changes in the properties and definitions given below are obvious, when this last requirement is omitted.

Suppose there is defined on $\mathfrak{P}$ a function $f(p)$. In defining a value for an integral $\int_E fdv$, we can follow the method of Young or that of Lebesgue.

For an analogue of the Young integration, we divide E into a finite or denumerable set of subsets E_n of $\mathfrak{E}$, and let m_n and M_n be the upper and lower bounds, respectively, of f on E_n. Let

$$S = \Sigma_n M_n v(E_n) \quad \text{and} \quad s = \Sigma_n m_n v(E_n).$$

Then the upper integral $\overline{\int}_E fdv$ is the greatest lower bound of the values of S and $\underline{\int}_E fdv$ the least upper bound of the values of s for all possible divisions of E into a finite or denumerably infinite number of sets belonging to $\mathfrak{E}$. We say that $\int_E fdv$ exists, when the upper and lower integrals are equal.

If f is not bounded on E, then the divisions of E must be restricted to be such that M_n and m_n are finite. In particular a necessary and sufficient condition for the existence of finite values for $\overline{\int}$ and $\underline{\int}$, the upper and lower integrals, is that there shall be at least one division of E into a denumerable set of distinct sets E for which the value of S formed for $|f|$, $\Sigma_n M_n v(E_n)$, is finite.

On the other hand we are able to give a definition of integration which is analogous to that of Lebesgue. We say that f is *measurable relative to* $\mathfrak{E}$, if the set E for which $f > l$ belongs to $\mathfrak{E}$ for every value of l. It follows from this that the sets E for which $l_1 \leqq f < l_2$ belong to $\mathfrak{E}$ for every l_1 and l_2.

Let us divide the interval $-\infty$ to $+\infty$ by means of the points $\cdots, l_{-n}, \cdots, l_{-1}, l_0, l_1, \cdots, l_n, \cdots$, and let E_n be the set for which $l_{n-1} \leqq f < l_n$. We say that f is *summable relative to v and* $\mathfrak{E}$, if there exists a division of $-\infty$ to $+\infty$ for which $\sum_{n=-\infty}^{n=+\infty} l_n v(E_n)$ is absolutely convergent.

If f is measurable and summable relative to v and $\mathfrak{E}$, then we define

$$(L) \int_E f dv = \lim_{d \doteq 0} \Sigma_n \lambda_n v(E_n),$$

where d is the maximum difference $l_n - l_{n-1}$, and $l_{n-1} \leqq \lambda_n < l_n$, which limit can be shown to exist.

These two definitions of integration, the $(Y) \int_E f dv$ and the $(L) \int_E f dv$, are not in general equivalent, the first of the two being the more inclusive in that it may define an integral for functions which are not integrable according to the second definition. For instance, if we suppose that $\mathfrak{P}$ is the linear interval (a, b), and the class $\mathfrak{E}$ is the class of all Borel measurable sets of points, then the (Y) integral method of definition will make all Lebesgue summable functions integrable, while the (L) integral method is restricted to Borel measurable functions. If however we assume that this $\mathfrak{E}$ is extended to include all the Lebesgue measurable sets, i. e., if we add to the class $\mathfrak{E}$ all the sets E for which there exists a Borel measurable set B_1 including it and a Borel measurable set B_2 included within it, for which

$$\text{meas } B_1 = \text{meas } B_2,$$

then the two definitions are equivalent.*

A similar result holds in the more general situation. If we extend the class $\mathfrak{E}$ so as to include all sets E for which there exists an E_1 and an E_2 belonging to $\mathfrak{E}$ such that $E_1 \leqq E \leqq E_2$, and $v(E_1) = v(E_2)$, and call the resulting class *complete as to* v, since no further extension will be possible, we shall have the proposition:

If $\mathfrak{E}$ *is complete as to* v, *then the general Young integral definition is equivalent to the general Lebesgue integral definition, and the integrals of* f *with respect to* v *on any set* E *of* $\mathfrak{E}$ *take the same value.*

There is no difficulty in writing down some of the more important properties of these integrals, either from the corresponding properties of the Stieltjes or the Lebesgue integral. We note only the following:

(1) If $E_1, E_2, \cdots, E_n, \cdots$ belong to $\mathfrak{E}$, and have no common

* Cf. for instance, de la Vallée Poussin (24), pp. 31–33.

elements, and if $\int_{E_n} fdv$ exists for every n, then if $E = \Sigma_n E_n$,

$$\Sigma_n \int_{E_n} fdv = \int_E fdv.$$

(2) If the f_n are defined on $\mathfrak{P}$ and $\lim_n f_n = f$ on $\mathfrak{P}$, then if the f_n are integrable relative to v on E for every n, and f is summable on E, f will be integrable relative to v and

$$\lim_n \int_E f_n dv = \int_E fdv,$$

if (a) the sequence f_n is monotonic non-decreasing, or (b) $|f_n - f|$ is a bounded function on E and with respect to n.

7. *Special Cases of the Fréchet Integral.*—(Cf. Young (29); Radon (19), pages 1305 ff.) We have already indicated in the preceding paragraph that if we assume $\mathfrak{P}$ to be any linear interval (a, b) and $\mathfrak{E}$ to be the class of all Lebesgue measurable sets of points on (a, b) while $v(E) = \text{meas } E$, the Fréchet integral reduces to the Lebesgue or Young integral, depending upon which manner of definition is followed out. Similarly if $\mathfrak{P}$ is any set of points on (a, b), and $\mathfrak{E}$ is the class of all sets measurable relative to $\mathfrak{P}$, then if we put $v(E) = \overline{\text{meas}}\, E$, we obtain the Pierpont integral for any set E of the class $\mathfrak{E}$.

If we let $\mathfrak{P}$ be the interval (a, b), and $v(E)$ the function of sets defined from the monotonic non-decreasing function $v(x)$ in § 5, and $\mathfrak{E}$ the class of all sets of points measurable relative to $v(x)$, which class will be complete as to v, then we get an extension of the Lebesgue integral to the Stieltjes integral, which, by putting $v(x) = x$, reduces to the ordinary Lebesgue integral, and has properties analogous to those of both the Stieltjes and Lebesgue integrals.*

Young has given an extension of the Stieltjes integral on the basis of monotonic sequences of semi-continuous functions which is equivalent to the one which we have just derived. He defines the value of $\int_a^b fdv$, v being a monotonic non-decreasing function, for functions which are upper or lower semi-continuous but constant on subintervals of the interval (a, b), i. e., what he calls simple u- and l-functions. If $f(x)$

* Cf. also Daniell, this BULLETIN, vol. 23, pp. 209, 211.

is constant in each of the intervals formed by the points of division $a = x_0, x_1, \cdots, x_n = b$, then his expression can be reduced to

$$\int_a^b f dv = \sum_{i=0}^{n} f(x_i)(v(x_i + 0) - v(x_i - 0)) \\ + \sum_{i=0}^{n-1} f(x_i + 0)(v(x_{i+1} - 0) - v(x_i + 0)),$$

where we assume that $v(x_0 - 0) = v(a - 0) = v(a)$ and $v(x_n + 0) = v(b + 0) = v(b)$.

From this definition of integration for simple upper and lower semi-continuous functions he passes by monotonic sequences of these functions to the integrals of upper and lower semi-continuous functions, and then proceeds in the same way to the general case, i. e., he formulates the following definition:

Form the integrals with respect to $v(x)$ for all upper semi-continuous functions less than the given function, and take the upper bound of these integrals; form the integrals with respect to $v(x)$ of all lower semi-continuous functions greater than the given function and take the lower bound of these integrals; if these two bounds agree the function is said to have an integral with respect to $v(x)$, which is the common value of the bounds.

In showing the equivalence of this definition to the one which we have proposed above, we note first that the value of the integral of simple u- and l-functions as given by Young is the same as that which we should obtain from our definition. The fact that if a function is integrable according to this Young definition, it is also integrable as above, with respect to $v(x)$, is then immediately evident.

To prove the converse, it is sufficient to show that if E is any set measurable relative to $v(x)$, and $f(x)$ has the value unity for x on E and zero elsewhere, then the $(Y)\int_E f dv$ exists, and is equal to $v(E)$. For this purpose we may use the property of $v(E)$ noted previously:

If $v(E)$ exists, then for every $\epsilon > 0$ there exists a closed set E' contained in E such that $v(E') - v(E) < \epsilon$;

and the further fact apparent from the definition of $v(E)$:

If $v(E)$ exists, then for every $\epsilon > 0$, there exists a set of non-overlapping open intervals α_n such that every point of E is interior to some α_n and $\Sigma_n v(\alpha_n) - v(E) < \epsilon$.

Radon has shown how one can define a $v(E)$ with respect to monotonic functions in space of any number of dimensions. So far however none but trivial examples for a general function space have been given. It still remains therefore to give examples of functions $v(E)$ for spaces of infinitely many dimensions, and function spaces, which are not trivial.

It may be of interest to note, finally, that the general Fréchet-Stieltjes integral of a bounded function, when existent on a complete class $\mathfrak{E}$, is expressible as an ordinary Stieltjes integral in the form $\int_l^L y d\mu(y)$, where $l < f < L$ and $\mu(y)$ is the value of $v(E_y)$ where E_y is the set of elements for which $f < y$. This integral in turn is expressible as an ordinary Riemann integral. In a sense then the general Fréchet integral is equivalent to the Stieltjes integral on a linear interval. The extension to the case of an unbounded function is obvious.

IV. The Hellinger Integral.

1. *Definition of the Hellinger Integral and its Relation to the Lebesgue.*—(Cf. Hellinger (11), pages 236 ff.; Hahn (7), pages 170–183; Riesz (20), page 462.) We conclude our discussion of definitions of integration with a brief mention of the Hellinger integral and its generalizations. It is closely related to the Stieltjes integral, as a matter of fact extends in a way the ideas underlying this latter integral, and also had its origin in an attempt to break up into component parts a Stieltjes integral found by Hilbert in his work on quadratic forms in infinitely many variables. It is defined as follows:

Let $v(x)$ be a monotonic non-decreasing continuous function of x in an interval (a, b). Further let $f(x)$ be any continuous function of x, which is constant in the intervals in which $v(x)$ is constant, i. e., if $v(x_2) - v(x_1) = 0$, then $f(x_2) - f(x_1) = 0$. Divide the interval (a, b) into any finite number of intervals by means of the points $a = x_0, x_1, \cdots, x_n = b$, and form the sum

$$\sum_{i=0}^{n-1} \frac{(f(x_{i+1}) - f(x_i))^2}{v(x_{i+1}) - v(x_i)},$$

the quotient being defined to be zero when $v(x_{i+1}) = v(x_i)$. It can then be shown with the aid of the Schwarz inequality that this *sum does not decrease when we subdivide the intervals* (x_i, x_{i+1}). *The least upper bound of this sum for all possible divisions of the interval* (a, b) *is defined to be the Hellinger integral*

$$\int_a^b \frac{(df)^2}{dv}.$$

If the Hellinger integrals

$$\int_a^b \frac{(df_1)^2}{dv} \quad \text{and} \quad \int_a^b \frac{(df_2)^2}{dv}$$

exist, then it can be shown that the Hellinger integral

$$\int_a^b \frac{df_1 df_2}{dv}$$

also exists and that we have

$$\left(\int_a^b \frac{df_1 df_2}{dv}\right)^2 \leqq \int_a^b \frac{(df_1)^2}{dv} \int_a^b \frac{(df_2)^2}{dv},$$

which corresponds to the Schwarz inequality.

Hahn has stated a necessary and sufficient condition under which the Hellinger integral of f with respect to v will exist, and incidentally given the relation between Hellinger and Lebesgue integrals. Suppose we take the inverse $x(v)$ of the monotonic function $v(x)$. This will not have a unique definition at the values of v for which $v(x)$ is constant, but when we substitute it in the continuous function $f(x)$ which is constant where $v(x)$ is constant, we obtain a continuous function $F(v)$ of v defined in the interval $(v(a), v(b))$. Then Hahn's theorem is:

A necessary and sufficient condition for the existence of the Hellinger integral

$$\int_a^b \frac{(df)^2}{dv}$$

is that $F(v)$ *be the indefinite integral of a function* $F'(v)$, *which is summable and of summable square. Moreover we have*

$$\int_a^b \frac{(df)^2}{dv} = \int_{v(a)}^{v(b)} (F'(v))^2 dv.$$

This theorem thus expresses the Hellinger integral in terms of a Lebesgue integral, and a Lebesgue integral in terms of a Hellinger. For the proof of the theorem we refer to Hahn's memoir (7). Riesz previously obtained a similar result for an integral of the form

$$\int_a^b \frac{|df|^p}{(dv)^{p-1}}, \qquad (p > 1),$$

viz., a necessary and sufficient condition that a continuous function $F(x)$ be the indefinite integral of a function $F'(x)$ for which $\int_a^b |F'(x)|^p dx$ with $p > 1$ exists, is that there exist a finite upper bound for the sum

$$\sum_{i=1}^{n} \frac{|F(x_i) - F(x_{i-1})|^p}{(x_i - x_{i-1})^{p-1}}$$

for all possible subdivisions of the interval (a, b) into a finite number of subintervals.

So far the Hellinger integral has shown itself of value in the theory of quadratic forms in infinitely many variables and the related fields, in which it is effective in breaking up such a form into the sum of squares of linear forms in infinitely many variables, thus completing the analogy with the finite case. No simple reduction of this character has been thus far made by using the Lebesgue integral only, even though the Hellinger integral is expressible in terms of the Lebesgue.

2. *Generalizations of the Hellinger Integral.*—(Cf. Radon, pages 1351 ff.; the Moore generalization has been given by Moore in lectures at the University of Chicago, 1915–17.) An extension of the Hellinger integral has been given by Radon. He presupposes a class $\mathfrak{E}$ of sets E of the type discussed in §§ 5 and 6 of Chapter III in connection with the Fréchet general integral, but limited to points in a space of a finite number of dimensions. He assumes further a function $v(E)$ which is monotonic, i. e., $v(E) \geqq 0$ for every E of the class $\mathfrak{E}$, and is absolutely additive, i. e., for every denumerable infinity of mutually distinct sets $E_1, E_2, \cdots, E_n, \cdots$ of $\mathfrak{E}$

$$v(\Sigma_n E_n) = \Sigma_n v(E_n).$$

He further assumes that the functions f are defined on the class $\mathfrak{E}$, are absolutely additive and so of bounded variation, and in addition such that if $v(E)=0$ then $f(E)=0$. A set E is then divided into a finite number of subsets $E_1, \cdots, E_n$ of the class $\mathfrak{E}$, and the expression

$$\sum_{i=1}^{n} \frac{|f(E_i)|^p}{(v(E_i))^{p-1}}$$

formed, the quotient being defined to be zero if $v(E_i) = 0$. Then the least upper bound of this expression for all possible finite subdivisions of the set E into sets E_i is defined to be the integral

$$\int_E \frac{|df|^p}{(dv)^{p-1}}.$$

When $p = 2$, this integral reduces to the Hellinger integral if $v(E)$ is formed on the basis of a continuous monotonic non-decreasing function as in § 5 of Chapter III; and to the sum of squares if formed on the basis of a monotonic non-decreasing function, constant except for a denumerably infinite set of points of discontinuity.

Radon proves also a theorem for the space of n dimensions, which corresponds to the Hahn theorem on Hellinger integrals:

A necessary and sufficient condition that the generalized Hellinger integral

$$\int_E \frac{|df|^p}{(dv)^{p-1}}$$

exist, is that there exists a function F defined on the fundamental set $\mathfrak{P}$, such that $f(E) = \int_E F dv$ for every E of $\mathfrak{E}$, and for which the generalized Stieltjes integral $\int_E |F|^p dv$ exists, this last integral being equal to the generalized Hellinger integral of f with respect to v.

The function F which appears in this theorem is of the nature of a derivative of the function $f(E)$ with respect to the monotonic function $v(E)$; in other words, this connects with the idea of the derivatives of functions with respect to monotonic functions or more generally functions of bounded variation. Some results in this direction have recently been obtained by Young,* though there still is undoubtedly a considerable field for investigation.

A generalization of the Hellinger integral in a different direction has been proposed by Moore. He starts from the integral in the form

$$\int_a^b \frac{df_1 df_2}{dv}$$

and observes that it is bilinear in f_1 and f_2, and it is the bilinear aspect of the Hellinger integral which dominates his generalization. The germ of the generalization is contained in the observation that the Hellinger integral

$$\int \frac{df_1 df_2}{dv}$$

can be written as the limit (or the least upper bound if $f_1 = f_2$) of a double sum of the type

$$\sum_{i=1}^{n} \sum_{j=1}^{n} f_1(s_i)\omega_\pi(s_i, s_j)f_2(s_j),$$

where $a \leqq s_1 < s_2 \cdots < s_n \leqq b$ is a partition π of the interval (a, b), and $\omega_\pi(s, t)$ is a function of two variables which depends for its value on the values of v and the partition π.

Suppose then that $\mathfrak{P}$ is a class or set of elements p, concerning the character of which nothing is postulated, i. e., they are perfectly general. We denote by σ any finite collection $(p_1, \cdots, p_n)$ of distinct elements of $\mathfrak{P}$. Let there be defined functions $\xi(p)$ which make correspond to every value of p a real or complex number a. We denote by $\bar{\xi}$ the conjugate function, i. e., the function which for every p takes on the values conjugate to those of ξ.

Let $\epsilon(p, q)$ be a function of the two variables p, q, each of which varies over the range $\mathfrak{P}$. We shall assume that $\epsilon(p, q)$ has the following two properties:

(*a*) $\epsilon(p, q) = \bar{\epsilon}(q, p)$.

(*b*) For any σ the determinant of the values $\epsilon(p_i, p_j)$, $i, j = 1, \cdots, n$ is positive and not zero.

* Cf. *Proceedings Lond. Math. Society*, ser. 2, vol. 15 (1916), pp. 35–64. See also de la Vallée Poussin: (24), pp. 67 ff., where the case in which $v(E) = \text{meas } E$, i. e., $v(x) = x$ is treated.

Then we define the operation J as follows:

$$J\bar{\xi}\xi = -\lim_{\sigma} \frac{\begin{vmatrix} 0, & \bar{\xi}(p_1), & \cdots, & \bar{\xi}(p_n) \\ \xi(p_1), & \epsilon(p_1, p_1), & \cdots, & \epsilon(p_1, p_n) \\ \cdot & \cdot & \cdot & \cdot \\ \xi(p_n), & \epsilon(p_n, p_1), & \cdots, & \epsilon(p_n, p_n) \end{vmatrix}}{\begin{vmatrix} \epsilon(p_1, p_1), & \cdots, & \epsilon(p_1, p_n) \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \epsilon(p_n, p_1), & \cdots, & \epsilon(p_n, p_n) \end{vmatrix}},$$

in which by

$$\lim_{\sigma} F(\sigma) = a,$$

we mean* that for every $e > 0$, there is a σ_e such that if σ contains σ_e, then $|F(\sigma) - a| \leqq e$.

More recently he has replaced this definition by one in which the least upper bound notion replaces that of limit as to σ. It is as follows:

Form for any σ the least upper bound of the values of

$$\sum_{i=1}^{n} \bar{\xi}(p_i)\eta(p_i) \sum_{i=1}^{n} \bar{\eta}(p_i)\xi(p_i)$$

for all functions η for which we have

$$\sum_{i,j=1}^{n} \bar{\eta}(p_i)\epsilon(p_i, p_j)\eta(p_j) = 1.$$

Then the least upper bound of these least upper bounds for all possible σ is defined to be $J\bar{\xi}\xi$. These two definitions are equivalent, i. e., for every function ξ they yield the same positive finite† or infinite value.

We notice that this operation J is essentially dependent upon the ϵ chosen in any particular situation. If we specify our ϵ we have theoretically determined the operation J, and the functions ξ for which the $J\bar{\xi}\xi$ exists, i. e., is finite. For instance if $\mathfrak{P} = (1, 2, \cdots)$ and $\epsilon(p, q) =$ the Kronecker

* Cf. *Proceedings Nat. Acad. Sciences*, vol. 1 (1915), p. 630.

† As an instance of a function $\xi(p)$ for which $J\bar{\xi}\xi$ is finite, we might note $\epsilon(p, q)$ for q fixed. For we have

$$J\bar{\epsilon}(p, q)\, \epsilon(p, q) = \epsilon(q, q).$$

δ, i. e., zero for $p \neq q$ and unity for $p = q$, then

$$J\bar{\xi}\xi = \sum_{p=1}^{\infty} \bar{\xi}(p)\xi(p)$$

and the functions on which J operates are those for which the sum of squares of absolute values is convergent. More generally if the class $\mathfrak{P}$ is perfectly general, and $\epsilon(p, q)$ is as above, i. e., zero for $p \neq q$ and unity for $p = q$, then the functions ξ for which $J\bar{\xi}\xi$ exists are different from zero only at most at a denumerable set of elements $p_1, \cdots, p_n, \cdots$ and

$$J\bar{\xi}\xi = \Sigma_n \bar{\xi}(p_n)\xi(p_n).$$

The elements $p_1, \cdots, p_n, \cdots$, at which the functions for which $J\bar{\xi}\xi$ exists are different from zero, may differ for different functions. If $\mathfrak{P}$ is the linear interval (a, b) this will then give an operation for the class of functions which are different from zero only at a denumerable set of points, which functions are disregarded in Lebesgue integration.

If $\mathfrak{P}$ is the interval $(a < p \leqq b)$ and

$$\epsilon(s, t) = \begin{matrix} s - a \text{ if } s \leqq t \\ t - a \text{ if } s \geqq t, \end{matrix}$$

then

$$J\bar{\xi}\xi = \int_a^b \frac{d\bar{\xi}d\xi}{dp},$$

i. e., the J-operation reduces to the Hellinger integral and the functions ξ are the continuous functions for which a Hellinger integral exists, i. e., according to the theorem of Hahn (§ 1) the continuous functions which are the indefinite integrals of functions which are summable and have summable squares.

Similarly the Hellinger integral

$$J\bar{\xi}\xi = \int_a^b \frac{d\bar{\xi}d\xi}{d\psi}$$

is obtained if

$$\epsilon(s, t) = \begin{matrix} \psi(s) - \psi(a) \text{ for } s \leqq t \\ \psi(t) - \psi(a) \text{ for } s \geqq t, \end{matrix}$$

where ψ is a properly monotonic increasing function on the interval $a < p \leqq b$.*

Moore has given non-trivial instances of ϵ's in the case in which p and q range over functional spaces. For instance if $\mathfrak{P}$ is the class of all continuous functions $\varphi(x)$ on the interval (a, b) then

$$\epsilon(\varphi_1, \varphi_2) = e^{\int_a^b \varphi_1(x)\varphi_2(x)dx}$$

has the properties required of the ϵ above.† Just what the character of the operation J is in this case does not seem to have been determined.

This general operation had its origin in the desire to obtain for a general range, functional or otherwise, a theory which would be a generalization of Hilbert's theory of biquadratic forms in infinitely many variables, as simplified by Hellinger. In addition to accomplishing this and incidentally throwing light on what is essential in the Hellinger theory, interest undoubtedly attaches to this generalization as being an instance of a bilinear operation in a general situation with instances of a non-trivial character, and for this reason it is bound to be the subject of further consideration and attention.

The treatment of Borel's definition of integration on page 132 ff. is not entirely clear nor accurate. The definition should be interpreted as follows:

$f(x)$ is (B) integrable in case (a) there exists a set of singularities Z denumerable or even of measure zero, such that for every ϵ and for every set of intervals which has total length at most ϵ and is such that each interval of the set contains at least one point of Z, the Riemann integral of $f(x)$ on the complementary set P_ϵ exists, and (b) these Riemann integrals approach a finite limit as ϵ approaches zero. This limit is the (B) integral of $f(x)$ on (a, b).

Proposition (2) is not correct in that the condition given is necessary but not sufficient, an immediate consequence of proposition (3). If, in accordance with a suggestion recently

* Professor Moore informs me that he has recently succeeded in removing from the condition (b) above the hypothesis that the determinant $|\epsilon(p_i, p_j)|$ is not zero. This would allow ψ to be simply monotonic non-decreasing and the interval to be $a \leqq p \leqq b$, which is the case treated by Hellinger.

† Cf. *American Mathematical Monthly*, vol. 24 (1917), pp. 31 and 333.

made by Lusin,* we replace in condition (*a*) above the words "for every ϵ and for every set" by "for every ϵ there exists a set" and call the resulting integral a (*BL*) integral, then we can state proposition (2) in the following form:

A necessary and sufficient condition that $f(x)$ be (*BL*) integrable is that there exist a set of singularities Z, denumerable or of measure zero, such that (*a*) the Lebesgue integral of f on the set $Z+Z'$ exists, (*b*) if the (a_n, b_n) are the intervals complementary to $Z+Z'$ then for every n

$$\lim_{a_n' \doteq a_n,\ b_n' \doteq b_n} (R) \int_{a_n'}^{b_n'} f$$

exists and is finite, and this limit is defined to be the integral over (a_n, b_n), (*c*) if ω_n is the maximum value of

$$\left| \int_{a_n'}^{b_n'} f \right|$$

for $a_n \leqq a_n' \leqq b_n' \leqq b_n$, then $\Sigma_n \omega_n$ is convergent.

If both the (*B*) and the (*BL*) integrals exist for a set Z, then the values will be the same. Also if either the (*B*) or the (*BL*) integral exist for different sets Z_1 and Z_2 then the resulting integrals are the same.

In the case of (*BL*) integrability we have the result that every (*L*) integrable function is (*BL*) integrable, but not conversely, in as much as (*BL*) integrable functions may be non-absolutely integrable.

Finally we desire to remark that Borel's first definition of integration† can be interpreted in the sense of (B^*) integrability. The footnote on page 135 should be revised accordingly.

* Cf. *Annali di Matematica*, ser. 3, vol. 26 (1917) p. 113.

† Cf. *Comptes Rendus*, vol. 150 (1910) pp. 375–7.