Now we have exhausted our store of invective. Let us close as we began by saying that it is a clear and interesting little book, whose appearance we heartily welcome.

J. L. COOLIDGE.

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## NON-EUCLIDEAN GEOMETRY.

*The Elements of Non-Euclidean Plane Geometry and Trigonometry.* By H. S. CARSLAW. London, Longmans, Green and Company, 1916. 16mo. vii+179 pp.

MORE than a score of years ago a writer of fluent pen published a short article with the title "Is the Non-Euclidean Geometry Inevitable?" His decision was affirmative, his interpretation of the question being "Are the conclusions of non-euclidean geometry inevitable?" We have learnt that they are, but we have also learnt that both the conclusions and the consequences are unavoidable, and that books dealing with the subject must be expected with certainty and met with fortitude. The number already published is already large, as we see by Somerville's compendious bibliography,* and the desire to publish others is so strong that not even the war can choke it. But there is always room at the top, and if a newcomer does not succeed in getting there himself, he raises the existing leader so much the higher. The book before us is a good one. We shall cheerfully damn it in detail later on, let us first praise it in general.

To begin with it is an interesting book. The field covered is not wide. Some things are omitted which we regret to miss, but nothing is included which might better have been left out. The choice of material seems largely guided by the principle that the book was primarily written for teachers, and the author has not shut his eyes to the fact that the average Anglo-Saxon mathematical teacher is too busy to read a long mathematical book, and too weak scientifically to understand a deep one. In consequence of this, certain standard topics like the realization of non-euclidean geometry on surfaces of constant total curvature or the subtleties of the Cayleyan metric are passed over in silence. The most serious omissions come from the restriction to plane geometry. In euclidean

* London, Harrison, 1911.

geometry, the most interesting elementary facts about the geometry of three dimensions may be inferred by the light of nature from what happens in the plane, but in non-euclidean space there are exceptions to this convenient rule. For instance, there is nothing in the geometry of the elliptic plane that foreshadows the properties of Clifford's skew parallel lines in space. These parallels constitute one of the most novel and interesting figures in all of non-euclidean geometry; it is a pity that a reader whose knowledge is bounded by the present book must remain in ignorance of them. It is pleasant to note in this connection that the author has a keen didactic instinct. Whenever he is forced to give a proof that is long or difficult, he divides the work into stages of reasonable length, explaining clearly just what is accomplished in each stage.

The first, yes, and much the greatest difficulty with which the writer on non-euclidean geometry has to deal is that of the fundamental assumptions or axioms. The present writer has met this difficulty in exemplary fashion. He recognizes that it would be equally unwise to give a long analysis of axioms according to modern abstract principles, or merely to say that the axioms are Euclid's except the parallel axiom. What he does in fact amounts to taking Hilbert's system, but as all readers can not be presumed to be familiar with this, he puts the matter somewhat differently. Euclid's axioms are retained except the one about parallels, but they are pieced out by assuming the first congruence theorem for triangles, by Hilbert's axiom that a given line segment may be extended a given amount in either direction, Pasch's axiom that a line in the plane of a triangle which passes between two vertices, and does not pass between a second pair nor go through a vertex, must pass between the third pair, and lastly an elaborate continuity axiom for straight lines which, incredible dictu, is never explicitly used. All this is done in a very few pages and completed by certain new constructions for perpendiculars and bisectors. The remainder of the first two chapters is of a historical nature, and follows conventional lines.

The third chapter, dealing with hyperbolic plane geometry, is the strongest in the book. The author takes a legitimate pride in the fact that he nowhere uses continuity in this chapter, but builds with not a little skill on the remaining axioms. No construction is ever used whose possibility and correctness has not previously been shown. He makes good use of the formulas which connect the parts of a right triangle with those of a quadrilateral with three right angles, formulas which are fundamental in problems of construction in the hyperbolic plane, but which are given scant notice in many of the text-books. The only adverse comment we feel inclined to make is that in spots the logical structure is so very delicate, that an inattentive reader might suspect the existence of mistakes that are not there. The definition of "equidistant curve" on page 83 is incorrect, as it gives only half of such a curve. And thus we reach page 91 and the end of Chapter III and the middle of the book. It would have been a far more flawless piece of work if the author had written "Finis" at that point.

The fourth chapter deals with hyperbolic trigonometry. The author, contrary to the custom of certain other elementary writers, develops this by two-dimensional methods only, the fairy godmother that smoothes out all obstacles being the limiting curve or orocycle. This has been defined correctly on page 80 and on the following page we are told accurately what is meant by congruent orocyclic arcs. From this definition we may safely infer the meaning of a rational ratio of two such arcs "Right and jest, jest and right," as the immortal Disko Troop remarked. But we know nothing of incommensurable arcs on orocycles, nor do we yet know that an orocycle is a continuous curve, and when we read on page 93 "if the arcs are incommensurable we reach the same conclusion by proceeding to the limit," we feel as if we had been rudely awakened by the whole logical structure crashing down about our ears. The same seductive process of proceeding to the limit is used again on pages 128 and 139. It is so fatally easy! But why give at the outset an elaborate continuity axiom and why employ the logical rigor of the strictest sect of the pharisees during the whole first part of the book, if one is coming to this at the last? A similar lapse occurs in the handling of the equation*

$$\tanh a = \cos f(\alpha).$$

It is shown that $f$ decreases as $a$ increases, and it is assumed without further ado that it is continuous and differentiable!

* Pp. 106ff.

Chapter V deals with the differential of length in the hyperbolic plane, and measures of area. An area is taken as a primary concept and not further defined.

Chapter VI brings us at last to the elliptic geometry. It is brief and follows conventional methods. There is a slight slip on page 132, where the author makes use of the theorem that an exterior angle of a triangle is greater than either opposite interior angle. This is only true in the elliptic plane if the region be sufficiently small. On page 133, line 3, *FB* should read *FP*. This is the only printer's error which we have noticed in the book.

Chapter VII deals with the elliptic plane trigonometry, and is the most difficult chapter in the work. The method followed is that originally devised by Gérard, although in spots, as on page 140, the present author omits some tedious if essential details. The whole treatment suggests a didactic question of not a little interest to all who undertake to teach non-euclidean geometry. In developing the elementary parts of the subject, one may follow one of two different methods. The first is to develop a general geometry as far as possible, and to give the theorems characteristic of the particular geometries only after the general theorems have all been put in evidence. Similarly in trigonometry, a general set of formulas is derived suitable to all three classical geometries, and the distinction of one from the other depends on the value of the space constant. The second method consists in making full use in the case of each geometry of the features characteristic of that geometry. Limiting ourselves to the consideration of recent text-books we may say that Killing* and the reviewer† have followed the first method, while not only the present author, but Liebmann,‡ Manning,§ and Sommerville|| have followed the second. It is largely a question of ideal. The first method lays emphasis upon the points of similarity of the three geometries, the second emphasizes their points of distinction. The first method is shorter, as the fundamental equations have to be deduced but once, the second approximates more closely to an ideal which the late Gaston Darboux once explained to the reviewer in about these words:

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* Grundlagen der Geometrie, Paderborn, 1893, especially pp. 80ff.
† The Elements of Non-Euclidean Geometry, Oxford, 1909.
‡ Nicht-euklidische Geometrie, Second Ed., Leipzig, 1912.
§ Non-Euclidean Geometry, Second Ed., Boston, 1915.
|| The Elements of Non-Euclidean Geometry, London, 1914.

"It is the distinctive characteristic of geometry, not to have a general method, but to find in the problem itself the methods best suited to its solution."

Let us note in passing an unintentional pleasantry on page 143 where we read "SOs is acute."

Our book closes with a peculiarly interesting chapter, No. 8. Most writers on non-euclidean geometry feel the necessity of showing that the subject will really "work" by exhibiting examples of one or more geometrical systems which obey just the desired hypotheses. This is usually done by a discussion of the geometry on certain surfaces of constant curvature, but this procedure is rather blind to a reader who does not know differential geometry. The present writer departs from this precedent, and gives a bird's-eye view of the three geometries at once by building the geometry of what is called "nominal lines" and "nominal points." A nominal line is nothing more nor less than a euclidean circle with regard to which a chosen fixed point has a preassigned power, or, in the limiting case, a line through that point. The credit for that idea is given to Poincaré, "Science et Hypothèse," and the reviewer, for one, had always supposed until recently that the idea was entirely derived from the lamented French geometer. As matter of fact a good share of the credit should be given to an earlier and less-known writer, DePaolis, who exhibited in 1878 a one-to-two conformal transformation from the non-euclidean to the euclidean plane, where non-euclidean lines passed over into euclidean circles orthogonal to a fixed circle, the square of whose radius was positive, negative, or zero.* The difference between this and the Poincaré scheme is largely a question of phraseology. Let us explain briefly how the plan works.

We take a fixed point $O$ and a fixed number $k$, positive, negative, or zero. We define as a nominal line a line through $O$ or a circle with regard to which $O$ has the power $k$. If $k$ be positive, we take a fixed circle with center $O$ and radius $\sqrt{k}$; the nominal lines all cut it orthogonally, and two points determine a nominal line, unless they be inverse in the fixed circle. All points within, or on, this circle are defined as nominal points, and two nominal points will always determine just one nominal line; we have an excellent example of the

* "La trasformazione piana doppia, etc.," *Memorie della R. Accademia dei Lincei,* series 3, vol. 2 (1878).

hyperbolic plane. If $k$ be negative, the nominal lines are circles which meet a fixed circle with center 0 and radius $\sqrt{-k}$ in diametrically opposite points, or, in the limiting case, lines through $O$. Two points will determine a nominal line, unless one be the reflection in $O$ of the inverse of the other in this circle. We take as nominal points the points within the circle, and the pairs of diametrically opposite points on the circle; we have an admirable elliptic plane. Lastly, in the case where $k$ is zero we take as nominal lines the lines or circles through $O$, and as nominal points the totality of finite points except $O$. This gives a good example of the euclidean plane. With regard to distances, we may follow the author and define them Cayley fashion by the logarithms of certain cross ratios, but this involves rather higher mathematical considerations that have been introduced before, and breaks down entirely for the euclidean case. The author's treatment of this last case is simply lamentable. He says:*

*We define the nominal length of a nominal line as the length of the rectilinear segment to which it corresponds . . . the nominal length of a nominal segment is unaltered by inversion with regard to a circle of the system.*

What this definition means, we do not know, as there is no indication of just how a nominal line *corresponds* to a rectilinear segment. The conclusion would seem to be that the nominal length of a nominal line is defined as the length of either the arc or the chord of the circle which is that nominal line. But a moment's thought shows that neither of these is invariant under the inversions in question. This erroneous statement suggests, however, another way of putting the thing which covers all three cases.

Let two nominal segments be defined as "congruent" if they may be transformed into one another by a succession of inversions or reflections in nominal lines. It is easy to see that no segment is congruent to a part of itself. If a segment be split in two, we may define the process of addition by saying that the length of the sum shall be the sum of their lengths. It can then be shown how any length can be measured in terms of any other.

Let us point out, in conclusion, that the whole scheme of nominal lines, beautiful as it is, represents a change on the

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* P. 158.

author's part from the second to the first of those ideals in non-euclidean geometry which we dwelt on above.

J. L. COOLIDGE.

## CORRECTION.

SPEAKING of M. Nau's translation of Sebokht's reference to the Hindu-Arabic numerals, in the BULLETIN for May, 1917 (volume 23, page 366), I remarked that no report of the matter "seems as yet to have appeared in English." My attention has since been called to the fact that Professor L. C. Karpinski announced the discovery in *Science* for June 21, 1912. While this does not concern Mr. Ginsburg's valuable note on the work and influence of Sebokht, the correction as to the publication of the extract in English should be made.

DAVID EUGENE SMITH.

## NOTES.

THE June number (volume 18, number 4) of the *Annals of Mathematics* contains the following papers: "Fermat's last theorem and the origin and nature of the theory of algebraic numbers," by L. E. DICKSON; "The modified remainders obtained in finding the highest common factor of two polynomials," by A. J. PELL and R. L. GORDON; "Nomograms of adjustment," by L. I. HEWES; "Closed algebraic correspondences," by A. A. BENNETT; "The intersections of a straight line and hyperquadric," by J. L. COOLIDGE; "The relation between the zeros of a solution of a linear homogeneous differential equation and those of its derivatives," by W. B. FITE; "Conjugate planar nets with equal invariants," by L. P. EISENHART.

AT the meeting of the Edinburgh mathematical society on May 11 the following papers were read: By L. R. FORD: "A geometrical proof of a theorem by Hurwitz and Borel"; by D. G. TAYLOR: "Geometrical illustrations of cyclant substitutions."