THE APRIL MEETING OF THE AMERICAN MATHEMATICAL SOCIETY IN NEW YORK.

THE one hundred and ninety-first regular meeting of the Society was held in New York City on Saturday, April 28, 1917, extending through the usual morning and afternoon sessions. The attendance included the following twenty-seven members:

Mr. D. R. Belcher, Professor E. G. Bill, Professor E. W. Brown, Dr. Emily Coddington, Professor F. N. Cole, Professor Elizabeth B. Cowley, Professor L. P. Eisenhart, Dr. C. A. Fischer, Professor T. S. Fiske, Professor W. B. Fite, Professor W. A. Garrison, Professor O. E. Glenn, Dr. Olive C. Hazlett, Professor Dunham Jackson, Mr. S. A. Joffe, Professor Edward Kasner, Professor P. H. Linehan, Professor W. R. Longley, Professor R. L. Moore, Mr. G. W. Mullins, Dr. Alexander Pell, Professor R. G. D. Richardson, Dr. J. F. Ritt, Dr. Caroline E. Seely, Professor Oswald Veblen, Dr. Mary E. Wells, Mr. J. K. Whittemore.

Ex-President E. W. Brown occupied the chair at the morning session, being relieved at the afternoon session by Professor Kasner. The Council announced the election of the following persons to membership in the Society: Professor C. F. F. Garis, Union College; Professor F. J. Holder, University of Pittsburgh; Dr. V. H. Wells, University of Michigan; Professor W. L. Wright, Lincoln University, Pa. Six applications for membership in the Society were received.

Professor L. P. Eisenhart was reelected a member of the Editorial Committee of the *Transactions*, to serve until October 1, 1920. Professor E. R. Hedrick was appointed delegate of the Society to attend the inauguration of President Jessup of the State University of Iowa on May 11–12. A committee consisting of Professors Fite, Birkhoff, and Veblen was appointed to consider and report to the Council any measures which it may be desirable to take to increase the interest and efficiency of the New York meetings of the Society.

Committees were also appointed to consider the question of the legal incorporation of the Society and to prepare a list of nominations of officers and other members of the Council to be elected at the annual meeting in December.

The following papers were read at this meeting:

(1) Professor W. B. FITE: "The relation between the zeros of a solution of a linear homogeneous differential equation and those of its derivatives."

(2) Dr. SAMUEL BEATTY: "The inversion of an analytic function."

(3) Professor MAURICE FRÉCHET: "Relations entre les notions de limite et de distance."

(4) Professor O. E. GLENN: "A fundamental system of formal covariants mod 2 of the binary cubic."

(5) Professor LUIGI BIANCHI: "Concerning singular transformations B_k of surfaces applicable to quadrics."

(6) Professor J. E. ROWE: "The projection of a line section upon the rational plane cubic curve."

(7) Mr. L. B. ROBINSON: "On partial differential equations which define certain covariants."

(8) Mr. J. K. WHITTEMORE: "Kinematic properties of ruled surfaces."

(9) Dr. OLIVE C. HAZLETT: "On Huntington's set of postulates for abstract geometry."

(10) Mr. E. F. SIMONDS: "Differential invariants in the plane."

(11) Mr. JESSE DOUGLAS: "On certain two-point properties of doubly infinite families of curves on an arbitrary surface."

(12) Professor L. P. EISENHART: "Conjugate planar nets with equal invariants."

(13) Dr. ALEXANDER PELL: "Solutions of the differential equation $dx^2 + dy^2 + dz^2 = ds^2$ and their application."

(14) Dr. C. A. FISCHER: "On bilinear and n-linear functionals."

(15) Professor E. B. WILSON: "Classification of real strains in hyperspace."

(16) Professor F. H. SAFFORD: "Irrational transformations of the general elliptic element."

(17) Dr. J. H. WEAVER: "Some algebraic curves."

(18) Professor R. L. MOORE: "A necessary and sufficient condition that a sequence of simple arcs of specified type should be equivalent, from the standpoint of analysis situs, to a sequence of straight segments."

(19) Professor DUNHAM JACKSON: "Second note on the parametric representation of an arbitrary continuous curve."

(20) Professor DUNHAM JACKSON: "Roots and singular points of semi-analytic functions."

(21) Professor OSWALD VEBLEN: "Doubly oriented lines."

(22) Dr. G. M. GREEN: "The intersections of a straight line and a hyperquadric."

(23) Dr. F. W. BEAL: "On a congruence of circles."

(24) Professor G. A. MILLER: "Possible characteristic operators of a group."

(25) Professor R. D. CARMICHAEL: "Examples of a remarkable class of series."

(26) Dr. W. L. HART: "Note on infinite systems of linear equations."

Professor Fréchet's paper was communicated to the Society by Professor D. R. Curtiss, Professor Bianchi's through Professor Eisenhart. Mr. Simonds and Mr. Douglas were introduced by Professor Kasner. Professor Bianchi's paper was read by Professor Eisenhart, and the papers of Dr. Beatty, Professor Fréchet, Professor Rowe, Mr. Robinson, Professor Wilson, Professor Safford, Dr. Weaver, Dr. Green, Dr. Beal, Professor Miller, Professor Carmichael, and Dr. Hart were read by title.

Abstracts of the papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. Professor Fite's paper deals with differential equations of the second and third orders and with certain ones of general order. The results are analogous to those given in a paper under the same title in the June number of the *Annals of Mathematics*. There may be cited in particular a theorem setting forth conditions under which a solution of an equation of the third order cannot vanish more than twice between two successive roots of another solution.

2. Dr. Beatty's paper appeared in full in the May BULLETIN.

3. Professor Fréchet's paper deals with classes (L) of abstract elements for which the limit of a sequence is defined, and determines conditions to be added such that the class (L) will be a class (D) admitting a definition of distance, but preserving the convergence ideas already adopted. The paper will appear in the *Transactions*.

4. Professor Glenn gives, in this paper, a complete system of covariants mod 2 of a binary cubic form having arbitrary

coefficients. The system is finite. The methods of generation and proof of the completeness of the set of fundamental covariants (which includes also five pure invariants) are developed from the point of view emphasized in the author's paper in volume 17 of the *Transactions* (page 545).

5. A plane and a point on it constitute a *facette*, the point being called the *center*. The tangent planes to a surface and the respective points of contact afford ∞^2 facettes which may be said to constitute the surface. Suppose we have a surface S and with each of its facettes f we assume associated ∞^1 facettes f' in such a way that the center of each facette f' lies in the corresponding plane of f and its plane passes through the center of f. In this way we get ∞^3 facettes f' which ordinarily cannot be coordinated into ∞^1 surfaces S'. It will be all the more unusual if the latter circumstance is satisfied by the ∞^3 facettes f' obtained when S undergoes a deformation by flexure and the facettes f' are carried along in invariable relation with respect to the facettes f. Suppose that for S_0, one deform of S, all the centers of the facettes f' associated with each f lie on the line of intersection of the plane of f and a fixed plane π. Professor Bianchi has proposed and solved the problem: To find what must be assumed concerning S_0 and its relation with the fixed plane π in order that for each deform of S_0 the ∞^3 facettes can be coordinated into ∞^1 surfaces S'. He shows that S_0 must be a quadric; that when it is not a quadric of revolution the fixed plane must be a principal plane and that the planes of the facettes f', associated with each facette f of S_0, envelope a cone projecting from the center of f, the focal conic of S_0 lying in the given diametral plane; when it is a quadric of revolution, the fixed plane must be a meridian plane, and the planes must form two pencils whose axes are the joins of the center of f with the foci of the meridian conic. These geometrical configurations had been obtained by Professor Bianchi formerly in his researches on the transformations of the deforms of quadrics. The paper will be published in the *Transactions*.

6. Professor Rowe's paper appeared in full in the June BULLETIN.

7. Given a differential system

$$A_{11}\frac{\partial f}{\partial x_2}+A_{12}\frac{\partial f}{\partial x_3}+\cdots+A_{1n}\frac{\partial f}{\partial x_{n+1}}=0,$$

$$A_{21}\frac{\partial f}{\partial x_2}+A_{22}\frac{\partial f}{\partial x_3}+\cdots+A_{2n}\frac{\partial f}{\partial x_{n+1}}=0,$$

$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$

$$A_{m1}\frac{\partial f}{\partial x_2}+A_{m2}\frac{\partial f}{\partial x_3}+\cdots+A_{mn}\frac{\partial f}{\partial x_{n+1}}=0,$$

where

A_{i1} are functions of x_1 only,

A_{i2} are functions of x_1, x_2 only,

$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$

A_{in} are functions of $x_1, x_2, \cdots, x_n$, but not of x_{n+1}

$(i = 1, 2, \cdots, m)$;

Mr. Robinson shows that the singular points of such a system can always be determined in advance and that if the coefficients be polynomials the set can be integrated by a finite number of operations. The equations which define the covariants of systems of ordinary differential equations can always be reduced to such a system. The paper will be published in the next issue of the *Johns Hopkins Circular*.

8. In Mr. Whittemore's paper a study of some properties of ruled surfaces is made by reference of the surface to the moving triedral attached to any curve on the surface. Some properties of the general ruled surface are obtained, the most interesting of which is this: Any ruled surface may be generated by a radius fixed in a sphere whose center moves with unit velocity along the line of striction and which turns about the tangent to the line of striction with angular velocity equal to the reciprocal of the parameter of distribution of the surface; it follows that generators of two ruled surfaces, having the same line of striction and the same parameter of distribution, drawn through a point of the line of striction make a constant angle. Some particularly simple results are obtained for ruled surfaces whose parameters of distribution are equal to the radius of torsion of the line of striction.

9. In 1913, Professor Huntington* gave a set of postulates

* *Math. Annalen*, vol. 73, pp. 522–559.

for ordinary euclidean three-dimensional geometry, in terms of a class K of undefined elements, called "spheres," and the relation R of inclusion. At the time, he thought that the "general laws" were independent; but more recently, he noticed that Postulates 7 and 16 are redundant, and raised the question as to whether any others might be redundant.

Dr. Hazlett's paper shows that several other postulates are redundant, of which may be mentioned Postulates 4 and 6, which are special cases of Postulate 15. Furthermore, if Postulate 15 is satisfied, the vertices of a tetrahedron cannot be coplanar unless two or more coincide, and, in particular, the vertices of a triangle cannot be collinear unless two coincide. A slight change of wording, however, obviates this difficulty. Finally, the triangle transverse axiom (which Professor Huntington proved by the aid of the "general laws" 1–11 and the existence postulates E_1–E_3) is proved without the aid of any postulates beyond the first ten general laws.

10. Mr. Simonds' paper deals with the invariants under groups of point and contact transformations of differential configurations consisting of more than one regular analytic element. Previous papers bearing on this subject are P. Rabut: "Théorie des invariants universels," *Journal de l'Ecole Polytechnique,* 1898; E. Kasner: "Differential elements of the second order, etc.," *American Journal,* 1906, the latter pointing out an error in the former.

The following are the principal results of the present paper:

1. Under a finite continuous group of point transformations there are no *essential* invariants other than those of *two* elements.

2. (*a*) Under an infinite group of point transformations a single regular element has no invariants of order greater than zero. (*b*) There are, however, more general configurations having invariants. (*c*) In the case of certain imprimitive groups the simplest configurations having invariants present some exceptional features. (*d*) If λ_n be the smallest number of elements in the simplest configuration having invariants, $\lambda_n = 2n + 2$ for the entire group, $\lambda_n = n + 3$ for the area-preserving group, $\lambda_n = 3$ for the equilong group. The result for the entire group agrees with Kasner, and not with Rabut. Most of Rabut's results are wrong.

3. A set of theorems on contact transformations corresponding to 1 and 2 are obtained.

11. Mr. Douglas's paper belongs to what may be called "geometry in the neighborhood of a curve," as distinguished from the usual differential geometry which relates chiefly to the neighborhood of a point. Every question in this theory leads analytically to a problem in linear differential equations. This fact is a particular aspect of the principle of Darboux's "auxiliary equation" or Poincaré's "equations of variation," namely, that given an *arbitrary* system of differential equations, the solutions infinitely near to a given one are controlled by a system of *linear* differential equations.

In the present paper, a general doubly infinite curve family F on an arbitrary surface is studied with regard to certain properties involving two points a and b, a considerable distance apart. Let the curve C of the family F join a and b. Suppose at b an element of length $d\sigma$ placed normal to C, and its extremities joined to a by the curves C', C'' of F, these enclosing at a the infinitesimal angle $d\omega$. The ratio $d\sigma/d\omega$ is denoted by $V(a, b)$. When a and b are interchanged in the above construction the value obtained for the ratio $d\sigma/d\omega$ is denoted by $V(b, a)$. A general formula is established which expresses $V(a, b)/V(b, a)$ in terms of a certain integral taken along the curve C from a to b. One of the results flowing from this formula is that the doubly infinite curve families F for which everywhere $V(a, b) = V(b, a)$ are those for which the geodesic curvature is the same, at any given point, for all the curves of the family through that point. This is a generalization of a theorem of Levi-Civita to the effect that the geodesics themselves have the property $V(a, b) = V(b, a)$. A well-known theorem of Straubel on optical families is also generalized and found to hold for a wider category of curve families. The considerations of this paper have been extended to higher dimensions.

12. Koenigs has shown by geometrical considerations that the perspectives of asymptotic lines of a surface from a point on a plane form a conjugate net with equal invariants, and he observed, conversely, that such a planar net is always the projection of asymptotic lines on a surface. He stated that the converse problem is reducible to quadratures. Professor Eisenhart has shown that the methods of ordinary differential equations can be applied to this converse problem, and has found the coordinates of the surface in a form which is a generalization of the formulas of Lelieuvre.

13. In Dr. Pell's paper solutions of the differential equation $dx^2 + dy^2 + dz^2 = ds^2$ are obtained which lead to the representation of the coordinates of any curve as the sum of the coordinates of an arbitrary minimal curve and those of a curve lying on the minimal cone, whose arc is equal to the arc of the given curve. The necessary and sufficient conditions that the curve be algebraic are very simple. Some interesting results are easily obtained.

14. It has been proved by Fréchet that a bilinear functional $U[f(s), \varphi(t)]$ can be put in the form

$$\int\int f(s)\varphi(t)d_2u(s, t),$$

where $u(s, t)$ is regular in t, and after modifying the definition of the variation of a function of two variables he has proved that the variation of $u(s, t)$ is the least upper bound of the expression $|U[f, \varphi]|/mf\,m\varphi$, where mf and $m\varphi$ are the maxima of $|f(s)|$ and $|\varphi(t)|$. In the present paper Dr. Fischer has obtained the function $u(s, t)$ in a different way and proved it to be regular in both arguments and unique. Then the work can be extended by mathematical induction to n-linear functionals. In the last part of the paper it is proved that a homogeneous functional of the nth order can always be expressed as a multiple Stieltjes integral.

15. Professor Wilson shows that the algebraic method of Gibbs, which was used in an earlier communication (*Transactions of the Connecticut Academy of Sciences*, volume 14 (1907), pages 1–57) to classify dyadics or strains in hyperspace without regard to reality, affords a very easy means of carrying out the further classification with respect to reality. In addition to the tonics, the shears, and the tonic-shears, we find cyclotonics and cyclotonic-shears. The article will be printed in the *Journal of the Washington Academy of Sciences*.

16. Professor Safford continues in this paper a series of articles which have appeared in the BULLETIN and in the *Archiv der Mathematik und Physik*. These investigations are based upon a formula, published by G. G. A. Biermann and derived from Weierstrass's lectures, which expresses the solution of $[F'(x)]^2 = AF^4(x) + 4BF^3(x) + 6CF^2(x) + 4B'F(x) + A$ as an irrational function of $P(x)$.

17. If two lines l_1 and l_2 in the same plane pass through two fixed points, A and B, and rotate about A and B according to a definite law, the locus of the intersection of l_1 and l_2 will be a definite curve.

Dr. Weaver, by this means, has set up several curves and proved some of the relations existing between the fixed points and the curves.

In particular he has set up a curve C_n of degree n, which has at one of the fixed points an $(n - 1)$ point, and such that the polar of the other fixed point with respect to C_n gives C_{n-1}.

18. Suppose that: (a) k and l are two parallel lines, (b) $A_1, A_2, A_3, \cdots$ and $C_1, C_2, C_3, \cdots$ are two sequences of points lying on k and l respectively and having, as sequential limit points, the points A and B respectively, (c) $A_1B_1C_1$, $A_2B_2C_2$, $A_3B_3C_3, \cdots$, and ABC are simple continuous arcs no two of which have any point in common and all of which lie, except for their end points, entirely between k and l, (d) as $n \doteq \infty$ the arc $A_nB_nC_n$ approaches the arc ABC uniformly as its limit in the sense that if $\epsilon > 0$ there exists n_ϵ such that if $n > n_\epsilon$ then each point of $A_nB_nC_n$ is at a distance less than ϵ from some point of ABC.

Professor Moore proposes to show that a necessary and sufficient condition that such a set of arcs $A_1B_1C_1$, $A_2B_2C_2$, $A_3B_3C_3, \cdots$ may be equivalent from the standpoint of analysis situs to a set of parallel segments of straight lines is that for every positive ϵ there should exist a positive δ_ϵ such that if n is any positive integer and X and Y are points of the arc $A_nB_nC_n$ and the distance from X to Y is less than δ_ϵ then that part of the arc $A_nB_nC_n$ which lies between X and Y lies entirely within some circle of radius less than ϵ.

19. In a paper recently presented to the Society (see BULLETIN, volume 23, page 68), Professor Jackson gave a proof of the theorem, apparently due originally to Fréchet, that an arbitrary continuous curve can be represented parametrically in such a form that the coordinates do not remain simultaneously constant throughout any interval of values of the independent variable. In the present note he points out that the theorem is practically obvious in the case of a rectifiable curve, and, by means of a generalization of the process which defines the length of such a curve, obtains a considerably simplified proof for the general case.

20. In his second paper, Professor Jackson shows how certain theorems about roots and singular points of analytic functions of two or more complex variables, Weierstrass's theorem of factorization, for example, can be extended, with appropriate modifications, to functions which are analytic in one of their arguments and merely continuous in all together. The treatment is based primarily on the use of contour integrals.

21. A doubly oriented line is a line in a three-space associated with one sense class among its points and one sense class among its planes. Professor Veblen's note deals with a number of theorems on order relations which cluster about this notion. It will be published in the second volume of Veblen and Young's Projective Geometry.

22. In Dr. Green's note, a simple proof is given of a formula for the intersections of a straight line and a hyperquadric which Professor Coolidge derived by an entirely different method in a paper communicated to the Society at the last annual meeting.

23. Dr. Beal's paper is concerned with a congruence C of circles which satisfies the following conditions. Any circle of the congruence lies in the tangent plane of a surface S and its center is the point of tangency M of the plane with S. At every point P of each circle a line is drawn which makes an angle φ with the plane of the circle. The projection of this line on the plane of the circle makes an angle ψ with the tangent to the circle at P. For any displacements of S in the neighborhood of M the point P is to move at right angles to this line. The surface on which P lies is a surface S_1 of a single parameter family of surfaces and is called a transform of S. The radius R of the circle and the angles φ and ψ are functions of u and v, the curvature coordinates of S. If the three quantities R, φ, and ψ are not restricted, congruences C exist for any surface S. The necessary and sufficient condition that lines of curvature correspond on the surface S and the surface S_1 is that R be constant. When ψ is a constant the principal radii of curvature t_1 and t_2 of any surface S_1 satisfy a very simple relation free from the angle θ which determines a particular surface S_1. When φ is a constant any surface parallel to S has associated with it a congruence C for which