EQUILONG INVARIANTS AND CONVERGENCE PROOFS.

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THE writer has studied the invariants of a pair of analytic curves under the equilong group with the main object of throwing light on the corresponding question in the more important conformal geometry.* The two theories present many analogies, but are not connected by a strict principle of duality. The number of invariants and their orders turn out to be the the same, though the results have to be calculated independently.

In some questions, however, the two theories differ essentially, not only in the methods to be employed, but also in the results obtained. This is true, in particular, with regard to the convergence of the power series entering into the formal calculations. This question was left unsettled in the paper cited.

The principal object of the present paper is to complete the equilong theory by showing that the series in question are always convergent. It thus follows that the equality of the absolute invariants is a sufficient as well as a necessary condition for the equivalence of two pairs of curves. The method used is to reduce the question to one in differential equations† and then to apply certain existence theorems, for solutions at a singular point, due to Briot and Bouquet.

1. *Calculation of the Invariants.*

The equilong group of the plane consists of all contact transformations which convert straight lines into straight lines in such a way that the distance δ between the points of contact of any two curves on a common tangent remains in-

* See Conformal Geometry, Proceedings of the Fifth International Congress, Cambridge (1912), vol. 2, pp. 81–87.

† Such a reduction is impossible in the conformal theory. We have instead a functional equation, and in some cases, as Dr. Pfeiffer has recently shown, the formal solution is actually divergent. In the language of the paper cited above, invariant relations of infinite order are required in conformal equivalence, but not in equilong equivalence.

variant. If we use Hessian line coordinates u, v (where v is the perpendicular distance from the origin and u is the angle which the perpendicular makes with the initial line), the group may be written, in the notation of dual numbers,

$$(1)\qquad U + jV = \text{Function } (u + jv), \quad \text{where} \quad j^2 = 0,$$

or, in separated form,

$$(2)\qquad U = \varphi(u), \quad V = v\varphi'(u) + \psi(u),$$

where φ and ψ are arbitrary analytic functions.* If we change the sign of v, we obtain the improper equilong transformations which preserve the magnitude of δ but reverse its sense.

A single regular analytic curve has no invariants: it can always be reduced to the normal form $v = 0$, that is, the origin (considered as an envelope of lines).

Let us now consider two curves having a common tangent; this tangent we may assume to be the line $u = 0$, $v = 0$. One of the curves we may assume reduced to $v = 0$. The other is defined say by $v = f(u)$, where f is any power series without a constant term. In the second plane let the two curves be written in the form $V = 0$, and $V = F(U)$.

We then have to consider the subgroup of (2) which converts the point $v = 0$ into itself, and the line $u = 0, v = 0$ into itself. This is

$$(3)\qquad U = \varphi(u), \quad V = v\varphi'(u),$$

where $\varphi(u)$ is any power series beginning with the first power of u.

In terms of power series our problem is now as follows: When will the curves

$$(4)\qquad v = \alpha_1 u + \alpha_2 u^2 + \cdots,$$

$$(5)\qquad V = A_1 U + A_2 U^2 + \cdots$$

be equivalent under a transformation of the form

$$(6)\qquad \begin{aligned} U &= a_1 u + a_2 u^2 + a_3 u^3 + \cdots, \qquad (a_1 \neq 0), \\ V &= v(a_1 + 2a_2 u + 3a_3 u^2 + \cdots)? \end{aligned}$$

The requisite condition is expressed by the identity

* See Scheffers, *Math. Annalen*, 1905. The lines and curves considered are oriented.

$$(7)\quad (\alpha_1 u + \alpha_2 u^2 + \cdots)(a_1 + 2a_2 u + 3a_3 u^2 + \cdots)$$
$$= A_1(a_1 u + a_2 u^2 + \cdots) + A_2(a_1 u + a_2 u^2 + \cdots)^2 \cdots.$$

Equating coefficients, we find first $a_1\alpha_1 = a_1 A_1$. Since $a_1 \neq 0$, it follows that $\alpha_1 = A_1$. Hence α_1 is an absolute invariant. In fact α_1, in the curve (4), is the distance from the origin along the tangent $u = 0$, $v = 0$ to the point of contact with the curve. This verifies the invariance of δ, the tangential distance of two curves.

Equating coefficients of u^n in (7), we find an equation involving $a_1, a_2, \cdots, a_n$, the coefficient of a_n being

$$n\alpha_1 - A_1 = (n-1)\alpha_1 \qquad (n > 1).$$

Hence, if we assume $\alpha_1 \neq 0$, the equation can be solved for a_n. Hence no higher absolute invariants exist.

THEOREM I. *A pair of curves whose tangential distance δ is not equal to zero has no absolute invariant, under the equilong group, except δ.*

To show that two pairs of curves having the same δ are actually equivalent, it is of course necessary to show that the first of the series (6), whose coefficients a_n are calculated as described above, is convergent. This we shall do later.

We consider next the case $\delta = 0$, that is, the case where the two curves form a horn angle.* Let the order of contact of the two curves be $h - 1$, where h may be $2, 3, \cdots$. In our reduced form one of the curves is the point $v = 0$, so the other must be of the form

$$v = \alpha_h u^h + \alpha_{h+1} u^{h+1} + \cdots \qquad (\alpha_h \neq 0).$$

This takes the place of (4). In the second plane the curve (5) must take the same form, since order of contact is obviously an arithmetic invariant. The transformation (6) remains the same.

The first equation obtained from the identity (7) is now

$$\alpha_h = a_1^{h-1} A_h,$$

which determines a_1.† In the next equation the coefficient of

* See the author's paper cited above.

† If $h-1$ is even, and if α_h and A_h have opposite signs, the a_1 thus found will be imaginary. In this case we may apply a preliminary improper equilong transformation, say $U' = U$, $V' = -V$, which will change the sign of A_h. Hence we can always take a_1 to be real. The other coefficients $a_2, a_3, \cdots$ are found rationally, hence the transformation will be real.

a_2 is $(2 - h)\alpha_h$; hence we can solve for a_2 unless $h = 2$. In general, the nth equation determines a_n unless $h = n$. Hence for a given order of contact, that is, a given value of h (greater than unity), there is one and only one equation of the set which cannot be solved for one of the coefficients (the one with highest subscript) of the transformation. This particular equation, the hth in the list, together with the previous equations, will enable us to eliminate $a_1, a_2, \cdots, a_{h-1}$, thus giving a relation between the coefficients $\alpha_h, \cdots, \alpha_{2h-1}$ and $A_h, \cdots, A_{2h-1}$ of the two curves. This relation can be separated in the form

$$J_{2h-1}(\alpha_h, \alpha_{h+1}, \cdots, \alpha_{2h-1}) = J_{2h-1}(A_h, A_{h+1}, \cdots, A_{2h-1}),$$

where J is a certain rational function of its arguments. Hence we have an absolute invariant of order $2h - 1$.

THEOREM II. *Any horn angle, that is, a pair of curves touching each other, has one and only one equilong invariant. If the order of contact is* $h - 1$, *the order of the absolute invariant* J_{2h-1} *is* $2h - 1$.

If we allow h to take the value unity, the curves will *not* be in contact (we may call this contact of order zero), and the invariant J_1 is merely δ, the tangential distance. It is thus easy to restate Theorem II so as to include Theorem I.

In the case of simple contact, $h = 2$, the invariant J_3 is of third order and has the following geometric meaning (not restricted to the canonical form in which one of the curves is reduced to a point)

$$J_3 = \frac{r_1 \dfrac{dr_1}{ds_1} - r_2 \dfrac{dr_2}{ds_2}}{(r_1 - r_2)^2} = \frac{\dfrac{dr_1}{d\theta_1} - \dfrac{dr_2}{d\theta_2}}{(r_1 - r_2)^2}.$$

Here r_1 and r_2 denote the radii of curvature of the two curves of the horn angle; ds_1 and ds_2 denote the elements of arc; $d\theta_1$ and $d\theta_2$ denote the changes in the inclination of the tangent. The radii and their rates of change are of course taken at the vertex of the angle.*

* The analogous conformal invariant of a horn angle is (see first citation)

$$I_3 = \frac{\dfrac{d\gamma_1}{ds_1} - \dfrac{d\gamma_2}{ds_2}}{(\gamma_1 - \gamma_2)^2},$$

where γ denotes curvature.

2. *Convergence Proofs.*

If two pairs of curves have the same absolute invariant, then it is possible at least formally to find a power series for an equilong transformation converting the one pair into the other. To show that the series thus obtained is always convergent, we restate our equivalence problem (again in its canonical form) in terms of differential equations.

If $v = f(u)$ and $V = F(U)$ are to be equivalent under the transformation $U = \varphi(u)$, $V = v\varphi'(u)$, then

$$\varphi'(u) = \frac{F(\varphi(u))}{f(u)}.$$

Here φ is the unknown function. It will be convenient to replace u by x, and φ by y. Thus our differential equation is

$$\frac{dy}{dx} = \frac{F(y)}{f(x)} = \frac{A_1 y + A_2 y^2 + \cdots}{\alpha_1 x + \alpha_2 x^2 + \cdots}. \tag{8}$$

The curves will be equivalent if, and only if, this differential equation admits an analytic solution of the form

$$y = a_1 x + a_2 x^2 + \cdots \qquad (a_1 \neq 0). \tag{9}$$

The formal conditions are obtained from an identity which is obviously the same as (7) with u replaced by x. Hence we have a single condition on the coefficients in (8), namely, $J(\alpha) = J(A)$. The series (9) then formally exists, in fact there will always be ∞^1 such series, one of the coefficients (namely a_h) being arbitrary.

To show that the series obtained are convergent (that is, that the radius of convergence is greater than zero) we might use directly Cauchy's method of majorants; but this is unnecessary, since we can appeal to the following result due to Briot and Bouquet:* If an equation of the form

* Goursat, Cours d'Analyse, vol. 2 (second edition), pp. 503, 504. It is there shown that if the coefficient a_{01}, or b in Goursat's notation, is not a positive integer, there is one solution; if b is a positive integer, there will be either 0 or ∞^1 solutions. But divergent series never arise.

This is of course not true for differential equations of all forms. For example,

$$\frac{dy}{dx} = \frac{y - x}{x^2}$$

is satisfied formally by $y = x + x^2 + 2!x^3 + 3!x^4 + \cdots$, which is divergent for all values of x other than zero.

$$(10) \quad \frac{dy}{dx} = \frac{a_{10}x + a_{01}y + a_{20}x^2 + a_{11}xy + a_{02}y^2 + \cdots}{x},$$

can be solved formally by a power series

$$y = c_1x + c_2x^2 + \cdots,$$

this series will necessarily be convergent.

We discuss first the case where the tangential distance δ is not zero. Then $\alpha_1 = A_1 \neq 0$, so we may write (8) in the form

$$(11) \quad \frac{dy}{dx} = \frac{(A_1y + \cdots)(\alpha_1 + \alpha_2x + \cdots)^{-1}}{x} = \frac{y + R(x, y)}{x},$$

where R is a power series beginning with terms of the second degree in x and y. This equation is of the form (10). Hence convergence is assured.

We take next the case $\delta = 0$. If the order of contact is $h - 1$, the differential equation (8) becomes

$$(12) \quad \frac{dy}{dx} = \frac{A_hy^h + A_{h+1}y^{h+1} + \cdots}{\alpha_hx^h + \alpha_{h+1}x^{h+1} + \cdots}, \quad (\alpha_h \neq 0, A_h \neq 0, h > 1).$$

To reduce this to the Briot and Bouquet form, we make a change in the dependent variable, using the substitution

$$(13) \qquad y = (z + \lambda)x, \quad \text{where} \quad \lambda^{h-1} = \frac{\alpha_h}{A_h}.$$

The transformed equation is found to be

$$(14) \qquad \frac{dz}{dx} = \frac{(h-1)z + R(x, z)}{x},$$

where the power series R starts with terms of the second degree. This is of the form (10), the coefficient b or a_{01} being the integer $h - 1$. Hence z will be a convergent power series. By (13), the same will then be true of y.

THEOREM III. *The convergence of the series defining the equilong transformation is thus guaranteed in every case.*

In our discussion we have assumed one of the curves of the pair in each plane to be reduced to a point, the origin. We may pass directly from this canonical form to the general form, and state our final result as follows:

THEOREM IV. *If one pair of curves*

$$v = \alpha_1 u + \alpha_2 u^2 + \cdots,$$

$$v = \beta_1 u + \beta_2 u^2 + \cdots,$$

is to be equivalent, under the equilong group, to a second pair of curves

$$V = A_1 U + A_2 U^2 + \cdots,$$

$$V = B_1 U + B_2 U^2 + \cdots,$$

the necessary and sufficient condition is the equality of a single absolute invariant J, that is,

$$J(\alpha, \beta) = J(A, B).$$

If the order of contact of the curves of each pair (this is obviously an arithmetic invariant) is $h - 1$, the invariant J is of order $2h - 1$.

If $h = 1$ (curves not touching), J is the tangential distance δ. If $h = 2$ (simple contact), J is a combination of the radii of curvature and their rates of variation, as given above.

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THE INVERSION OF AN ANALYTIC FUNCTION.

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THE demonstration of the existence of the inverse of an analytic function is made to depend in the Weierstrass theory upon the power series representation of the function and in the Cauchy theory upon the Jacobian of the real and imaginary parts of the function with reference to the real and imaginary parts of the variable. The proof presented in the following pages finds its source in the Goursat conception of an analytic function and is related as to method to the theory of sets of points.

Suppose the function $w = f(z)$ exists and has a finite derivative at each point of a simply connected domain D.