In (5) we have the equations of two lines different for different values of t, and the locus of their intersection is the R^2. In (14) we have three concurrent variable lines, and the locus of their intersection is the R^3. Hence in general the method consists in finding the locus of the points of concurrence of n concurrent lines subject to the condition that this point of concurrence be on the R^n.

PENNSYLVANIA STATE COLLEGE,
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FORD'S STUDIES ON DIVERGENT SERIES AND SUMMABILITY.

Studies on Divergent Series and Summability. By WALTER BURTON FORD. Michigan Science Series, Volume II. New York, The Macmillan Company, 1916. xi + 194 pp.

DURING the past twenty years there has been an ever increasing interest in the study of divergent series and their applications. Naturally a coexistent phenomenon has been a very large expansion in the volume of literature on this subject. An idea of the amount of this expansion may be gathered from the bibliography of Professor Ford's book, which, while not exhaustive, contains a list of some two hundred books and memoirs (principally memoirs), of which all but about twenty have appeared from 1895 on.

Thus it has become more and more difficult for one who has not followed recent work on divergent series to ascertain readily the known results in a certain branch of that field or the methods that have proved fruitful in studying certain aspects of the subject. This is alike a handicap for the experienced research worker whose investigations in other fields have naturally led to a consideration of divergent series, and to the beginner in research who feels attracted toward the subject of divergent series and wishes to orient himself rapidly in the field in order to find the avenues that may lead to new results.

To both of these classes of readers, as well as to many others, Professor Ford's admirable work will undoubtedly prove a boon. It presents in clear and concise fashion the fundamental features of each of the two grand divisions of divergent series, namely, asymptotic series and summable series, and in certain subdivisions of each leads the reader to the present boundaries of knowledge, at the same time pointing the road to further advances. The appearance of such books, written by American mathematicians and published under the auspices of American universities, augurs well for the future development of mathematics in this country. All those who have this development at heart must feel grateful to Professor Ford and to the University of Michigan for the present contribution.

We turn now to a more detailed consideration of the book. As indicated in the previous paragraph it divides itself into two parts, one devoted to asymptotic series, the other to summable series. The first part comprises Chapters I–III, the second part Chapters IV and V.

In Chapter I certain fundamental theorems with regard to the Euler-Maclaurin sum formula* are derived, and their application to the study of asymptotic developments of functions is illustrated by a consideration of Stirling's series. This latter discussion is followed by a brief account of Poincaré's theory of asymptotic developments, which closes the chapter. The different theorems about the Euler-Maclaurin sum formula are concerned mainly with different forms for the remainder term in that formula, and are chosen from the point of view of their usefulness in obtaining asymptotic developments. It is certainly a convenience to have these important results collected in such compact and usable form.

Chapter II is devoted to the application of the general theorems of the previous chapter to the derivation of the asymptotic developments for a number of typical cases of fundamental importance. Two general classes of functions are considered, those defined by infinite products and those defined by infinite series. Beginning with rather simple examples, the writer proceeds by gradual steps to a point where he is able to treat the problem of obtaining the asymptotic development of the general integral function of order greater than zero. He also obtains a general theorem with regard to the asymptotic development of functions defined by power series whose coefficients satisfy a certain restrictive condition. The chapter as a whole forms an excellent intro-

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* Professor Ford follows Barnes in using the designation Maclaurin sum formula instead of the above. However, as Euler's priority of discovery is well known, this usage seems to be lacking in historical accuracy.

duction to the literature that deals with the problem of determining the asymptotic development of a given function. The illustrative examples are well chosen with a view to giving the reader an adequate idea of the methods that have been found useful in attacking that problem.

In Chapter III a brief account is given of the progress that has been made in obtaining the asymptotic solutions of linear differential equations and linear difference equations. No demonstrations are given here; the writer contents himself with stating the principal results, giving numerous references to the literature and indicating certain important problems in the field whose solution is yet to be obtained.

In Chapter IV the reader is introduced to the second class of divergent series, namely, summable series. The writer selects six of the earlier and more widely used definitions of the sum of a divergent series and discusses certain of their properties. The definitions selected are the well known Cesàro and Hölder arithmetic mean definitions, Borel's exponential method, his integral definition and a generalization of the latter, and finally Leroy's extension of Borel's integral definition. The discussion of these definitions begins naturally with a proof of their consistency with the definition of convergence, i. e., a proof that any convergent series will be summable by the given definition to the value to which it converges. The treatment of consistency is made more compact and lucid by first proving a general lemma, by means of which the consistency of the various definitions is readily established.

The author then proceeds to define what he calls the boundary value condition. He wishes to consider as summable only series

$$\sum_{n=0}^{\infty} u_n \tag{1}$$

for which the corresponding power series

$$f(x) = \sum_{n=0}^{\infty} u_n x^n \tag{2}$$

has a radius of convergence equal to unity, and then accept only definitions of the sum of (1) for which

$$s = \lim_{x \to 1-0} f(x).$$

He has already stated in the preface his belief that some limitation of this sort is desirable and timely in order to avoid confusion and inconsistency in the study of series.

On this point the reviewer is not in agreement with Professor Ford. It is of course highly probable that many definitions that satisfy the condition of consistency will not be very useful in the study of divergent series, and it may be found desirable at some future time to exclude certain definitions that do satisfy this condition. However, the reviewer does not think the time is ripe for such exclusion and he further believes that the limitations that the author prescribes for the definition of summability are too narrow. He does not share Professor Ford's opinion, expressed in a footnote in the preface, that the present situation in the theory of divergent series is closely analogous to the one which caused Cauchy and Abel to rule out divergent series from the domain of analysis. At that time there was a decided vagueness with regard to the sense in which a divergent series could be considered as having a sum, and a lack of precision in the various attempts to define a sum. Whatever definitions have been proposed recently have been stated with all the precision of modern analysis, and, as far as the reviewer is aware, have not been chosen in a pure spirit of arbitrariness. They have been selected, either from their probable usefulness in studying certain interesting types of divergent series, or as natural outgrowths of earlier definitions or of attempts to frame general theories of divergent series.

If any limitations on the definition of summability are eventually agreed upon by workers in the field of divergent series, it seems natural to the reviewer that they should not exclude any series to which we would wish to ascribe a sum, provided we have a method of summation that gives to such a series a sum that is generally useful. The limitations suggested by Professor Ford do exclude many such series, for under these limitations no power series could be regarded as summable outside of its circle of convergence. Yet such a series can be summed within the polygon of summability by Borel's integral definition to a value which is the analytic extension of the function defined within the circle of convergence. It is surely convenient to ascribe this value to the series, and since we can obtain the value by one of the standard methods of summation, it would seem logical to include such a series in the class defined as summable.

After defining the boundary value condition, the chapter under consideration continues with a discussion of the relationship to this condition of the six definitions mentioned above. Following this, conditions are obtained under which these definitions possess certain other properties. The author considers the effect on the summability and value of a series of adding terms to or subtracting terms from the beginning of the series, the nature of the series formed by adding corresponding terms of two summable series and that formed by taking the Cauchy product. The results are summarized on page 91. In the remaining sections of the chapter the writer discusses absolutely summable series (Borel), uniform summability, and a few important properties of series summable by the Cesàro (or Hölder) definition, or by Borel's integral definition.

Chapter V deals with the application of the theory of summability to certain developments in orthogonal functions, namely to Fourier's series and some of the allied developments. In this connection the convergence of the developments, as a special case of summability, is considered as well. While the author does not in general obtain results not previously known, he makes here a distinct contribution to the subject in that he develops a general theory of summability of developments of this type, out of which the results for particular developments may be obtained as special cases. This serves to unify the whole treatment and to bring into greater prominence the essential features of the problem. The method of treatment is an extension of Dini's discussion of the convergence of developments of this type.

The chapter is divided into four parts. In the first part the author discusses the simplest case, the convergence and summability of the ordinary Fourier's series. In connection with this discussion he makes clear to the reader what are the essential steps in building up a general theory that will apply to developments of this type. In the next part he develops a number of general theorems about the representation of arbitrary functions by definite integrals, which form the basis for this general theory. In the third part he shows how Cauchy's calculus of residues can be used to advantage in applying the theorems of the previous part to the developments in question. He also discusses certain general properties of the functions appearing in the terms of the developments, which are useful in studying their convergence or summability.

In the fourth and final part of the chapter we find a discussion of three different types of developments, namely a class of important sine developments,* developments in Bessel's functions and developments in Legendre's functions. All these discussions are carried through on the basis of the author's general theory. The results, while not always the most general that have been obtained, are broad enough to show the power of the general theory. There is just one point in which results sufficiently complete for the applications have not been secured. In discussing the developments in terms of Bessel's functions, no information is obtained with regard to the convergence or summability of the series at the origin, or its uniform convergence or summability in the neighborhood of the origin. Without information of this sort we are not able to carry through some of the applications of these developments to problems in mathematical physics. It is true that most discussions of the developments in Bessel's functions leave this same point unsettled, but it is obviously a matter of much importance, and any treatment that overlooks it is incomplete in an essential point.

In connection with his discussion of the developments in Bessel's functions Professor Ford has very properly called particular attention to the fact that the first adequate discussion of the convergence of the Bessel's developments for points in the interval $0 < x \leqq 1$ was given by Dini, since the latter's priority in this matter has not received due recognition from certain writers.†

Chapter IV concludes with a discussion of the convergence and summability of the developments in Legendre's functions. The results here, though not the most general that have been

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* These developments have in the successive terms of the series trigonometric functions of the form $\sin \lambda_n x$, where the λ's are the roots of the equation

$$z \cos z + p \sin z = 0.$$

† Cf. for example p. 374 of Whittaker and Watson's Modern Analysis, where the development theorem is stated and ascribed to Hobson. It is true that Hobson's result went beyond Dini's in that he considered functions having a Lebesgue integral and discussed uniform convergence. However, making allowance for a few errors in formulas, the result stated by Whittaker and Watson was fully established by Dini, and it is unfortunate that Dini's work should have been referred to by these writers in such a way as to emphasize the errors and overlook the substantial contribution.

obtained,* are complete in the sense that both convergence and summability are established under fairly broad conditions for the entire interval $(-1 \leqq x \leqq 1)$, thus including the points $x = -1$ and $x = 1$, which are exceptional in the same sense that the point $x = 0$ is for the developments in Bessel's functions.

At the beginning of this review two classes of readers to whom Professor Ford's book should prove very useful, have been mentioned. In concluding, a third class should be noted, namely those who are already actively engaged in research work in the field of divergent series. The literature in this field has grown so rapidly in such a short time that it is sometimes difficult, even for those who have made a point of following it, to locate readily a particular result. Professor Ford's references to the literature are on the whole very complete and have evidently been made the object of much care. In this way his book furnishes readier access, not only to the many valuable results it contains, but also to a still greater number of related results in the literature of divergent series.

A list of errata that have been noted is appended:

P. vii, eq. (1)	for	$\frac{B_3}{5.6} \frac{1}{x^6}$	read	$\frac{B_3}{5.6} \frac{1}{x^5}$
P. 5, l. 7	"	(5)	"	(6)
P. 8, l. 2	"	§ 5	"	§ 6
P. 10, l. 9	"	$\epsilon > h$	"	$\epsilon < h$
P. 12, l. 1	"	$\sum_{x-a}^{b-h}$	"	$\sum_{x=a}^{b-h}$
P. 22, footnote 27	"	$e^{-(1/x)}$	"	e^{-x}
P. 38, l. 22	"	$x = + \theta_y$	"	$x = y + \theta_y$
P. 50, eq. (48) and following eq.	"	$(-z)^{a-(1/2)-iy}$	"	$(-z)^{a-(1/2)+iy}$
P. 52, l. 9	"	$-\pi < \varphi < \pi$	"	$-2\pi < \varphi < 0$
P. 66, l. 8	insert the words *the given* before *equation*			
P. 93, fig. 6	for	λ	read	2λ
P. 149, eq. 116	"	$-\frac{1}{\sqrt{2\pi z}}$	"	$-\frac{i}{\sqrt{2\pi z}}$
P. 149, the eq. preceding (119)	"	$(2v-1)$ in exponents	"	$(2v+1)$
P. 155, ll. 15, 22 and 33	"	$\begin{cases} t > 0 \\ t < 0 \end{cases}$	"	$\begin{cases} t > \epsilon \\ t < \epsilon \end{cases}$
P. 162, last line of footnote	"	Nielson	"	Nielsen
P. 184, l. 5	"	Allerdice	"	Allardice

CHARLES N. MOORE.

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* In the opinion of the reviewer attention should have been called to Gronwall's very exhaustive discussion of the summability of the developments in Legendre's functions (cf. *Math. Annalen*, vol. 75 (1914), p. 321) in connection with the references to the literature in this chapter.